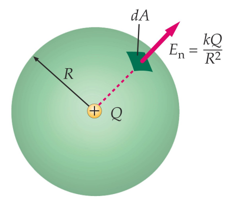
PHY204 Lecture 5 [rln5]

Electric Flux: Application (4)

Consider a positive point charge Q at the center of a spherical surface of radius R.
Calculate the electric flux through the surface.

- $\vec{E}$ is directed radially outward. Hence $\vec{E}$ is parallel to $d\vec{A}$ everywhere on the surface.
- $\vec{E}$ has the same magnitude, $E = kQ/R^2$, everywhere on the surface.
- The area of the spherical surface is $A = 4\pi R^2$.
- Hence the electric flux is $\Phi_E \doteq \oint \vec{E} \cdot d\vec{A} = EA = 4\pi kQ$.
- Note that Φ_E is independent of R.



tsl43

We begin this lecture with one more application of electric flux through a closed surface. This one is designed to lead right into our next topic.

The electric field $\vec{E}$ of a positive point charge Q is radial in direction, pointing outward. Its magnitude varies with distance by the inverse-square law.

We position that point charge at the center of a closed spherical surface of radius R as shown and calculate the electric flux through that surface.

There is no need to divide the surface into tiles because two conditions are satisfied: (i) The area vector of each tile has the same direction as the local electric field, namely radially outward, (ii) the field strength is the same everywhere on the surface.

The electric flux, therefore, is positive and its value is equal to the product of the area of the spherical surface and the electric-field strength at that radius:

$$\Phi_E = AE = (4\pi R^2)\left(\frac{kQ}{R^2}\right) = 4\pi kQ = \frac{Q}{\epsilon_0}.$$

The most striking feature of this result is that the flux is independent of the radius R. When we increase R, the area and the field strength change at reciprocal rates.

It can be shown, with some additional effort, that the electric flux remains the same if we position the charge Q elsewhere inside the spherical surface. However, if the we position Q outside the sphere, then the flux through the surface is identically zero.

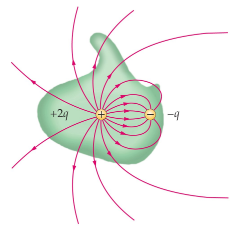
Gauss's Law for Electric Field

The net electric flux Φ_E through any closed surface is equal to the net charge Q_{in} inside divided by the permittivity constant ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{in} = \frac{Q_{in}}{\epsilon_0} \quad \text{i.e.} \quad \Phi_E = \frac{Q_{in}}{\epsilon_0} \quad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12} \mathrm{C^2 N^{-1} m^{-2}}$$

The closed surface can be real or fictitious. It is called "Gaussian surface".
The symbol $\oint$ denotes an integral over a closed surface in this context.

- Gauss's law is a general relation between electric charge and electric field.
- In electrostatics: Gauss's law is equivalent to Coulomb's law.
- Gauss's law is one of four Maxwell's equations that govern cause and effect in electricity and magnetism.



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On the previous page, we have, effectively, discovered a law of nature that is much more general than the situation might suggest. It is called Gauss's law for the electric field.

We have learned how to calculate electric flux through open or closed surfaces. In the context of Gauss's law, we are dealing with closed surfaces exclusively and name them *Gaussian surfaces*.

Pick a Gaussian surface of any shape or size, real or fictitious, and position it in a region of electric field. We express the flux through a closed surface symbolically by the expression,

$$\Phi_E \doteq \oint \vec{E} \cdot d\vec{A}.$$

Gauss's law states the electric flux through a closed surface only depends on the net charge inside:

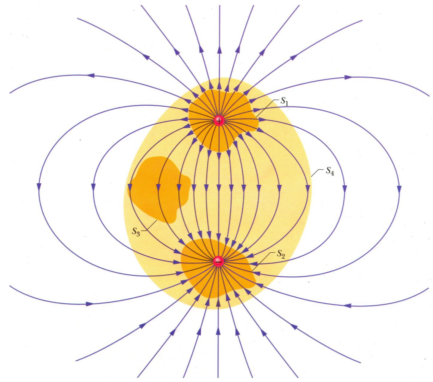
$$\Phi_E = \frac{Q_{in}}{\epsilon_0}.$$

By net charge we mean that, for example, that a proton and an electron add up to zero net charge.

Any charges that are positioned outside the surface do produce an electric field at the surface but their flux contributions through the surface always add up to zero.

Gauss's law is very general, as already mentioned. It holds even when charges are in motion or when there are electric fields generated by means other than electric charges (a topic to be discussed later).

Gauss' Law for Electric Field (Illustration)



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This slide shows an electric dipole, two particles with charges $+q$ and $-q$ separated some distance from each other. The strength and direction of the surrounding electric field is indicated by field lines.

Also shown in various shades of color are four closed surfaces, $S_1, \ldots, S_4$, here to be employed in the role of Gaussian surfaces.

If we were to calculate the electric flux Φ_E through each of these four surfaces, we would find the value $+q/\epsilon_0$ for S_1, $-q/\epsilon_0$ for S_2, and zero for S_3 and S_4.

The point to be emphasized is that we do not have to go through the trouble of actually performing that calculation, because Gauss's law gives us all four answers almost for free.

It is easy enough to see from the general direction of the electric field that the flux through S_1 is positive and the flux through S_2 negative, but it takes a rather elaborate numerical integration to came up with the values $\pm q/\epsilon_0$ for S_1 and S_2 and with zeros for S_3 and S_4.

Note that what matters in Gauss's law is the net charge Q_{in} inside. In the case of S_4, the net charge vanishes because $Q_{in} = q + (-q) = 0$.

Gaussian surface problem (1)

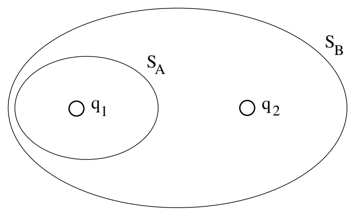
Two Gaussian surfaces S_A and S_B are shown in cross section.

Charge q_1 is on the inside of S_A and S_B.

Charge q_2 is on the inside of S_B only.

The electric fluxes produced by charges q_1 and q_2 through S_A and S_B are $\Phi_E^{(A)} = 5C/\epsilon_0$ and $\Phi_E^{(B)} = 3C/\epsilon_0$.

Find the electric charges q_1 and q_2.



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Suppose we have two particles with unknown charges q_1 and q_2 positioned as shown. Also shown in cross section are two Gaussian surfaces S_A and S_B. Surface S_A contains particle 1 only. Surface S_B contains both particles.

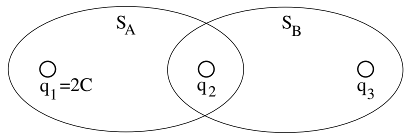
Somebody went through the trouble of measuring the electric flux through each surface, which turned out to be positive in both cases.

The fluxes are not expressed in the usual SI unit [Nm^2/C], but in the equivalent unit [C/ϵ_0], which has the advantage that it tells us directly what the charge inside the Gaussian surface is, namely $Q_A = 5\mathrm{C}$ inside S_A and $Q_B = 3\mathrm{C}$ inside S_B.

Since q_1 is the only charge inside S_A we thus conclude that $q_1 = Q_A = 5\mathrm{C}$. Given that both particles are inside S_B we infer that $Q_B = q_1+q_2$. The charge of the second particle must, therefore, be negative: $q_2 = 3\mathrm{C} - 5\mathrm{C} = -2\mathrm{C}$.

Gaussian surface problem (2)

The electric fluxes through the Gaussian surfaces S_A and S_B are $\Phi_E^{(A)} = 1C/\epsilon_0$ and $\Phi_E^{(B)} = 3C/\epsilon_0$, respectively.



Find the electric charges q_2 and q_3.

tsl46

In this variation of the same problem, we have three particles and again two Gaussian surfaces. Each surface contains two particles.

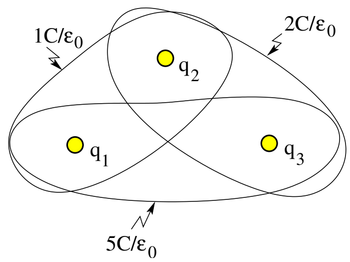
Given is the charge of the first particles, $q_1 = 2\text{C}$, and the electric flux through both surfaces, again in units which tell us directly the net charge each surface contains: $Q_A = 1\text{C}$ inside S_A and $Q_B = 3\text{C}$ inside S_B.

What are the values of the charges q_2 and q_3? We begin with surface S_A, which contain only one unknown charge. Gauss's law tells us that $Q_A = q_1 + q_2$, from which we conclude that $q_2 = -1\text{C}$.

Gauss's law also tells us that $Q_B = q_2 + q_3$ Having already determined q_2, we infer that $q_3 = 4\text{C}$.

Gaussian surface problem (4)

Three point charges q_1, q_2, q_3 produce electric fluxes through the three Gaussian surfaces as indicated.



(a) Find the net charge $Q = q_1 + q_2 + q_3$.
(b) Find the individual charges q_1, q_2, q_3.

tsl48

Here the charges q_1, q_2, q_3 of all three particles are unknown. What are their values if we are given the electric flux through three Gaussian surface that each contain two particles?

We are now sufficiently experienced that we do no longer need to name the surfaces. We can tell from what we see that

$$q_1 + q_2 = 1\text{C}, \quad q_1 + q_3 = 5\text{C}, \quad q_2 + q_3 = 2\text{C}.$$

We have three linear algebraic equations for three unknown. We can solve them in many different ways.

Here is one way: subtract the third equation from the second, then add the result to the first equation. We thus obtain $q_1 = 2\text{C}$. Substitution of this result in to the first and second equations then yields $q_2 = -1\text{C}$ and $q_3 = 3\text{C}$, respectively.

The result for part (a), which follows from part (b),

$$Q = 2\text{C} - 1\text{C} + 3\text{C} = 4\text{C},$$

can also be calculated without first calculating the charges q_1, q_2, q_3 individually.

We note that each particle is inside two of the three surfaces. If we add the three fluxes, then, according to Gauss's law, we are adding the contents of all three surfaces, which is twice the sum of all three charges. Therefore, we can write,

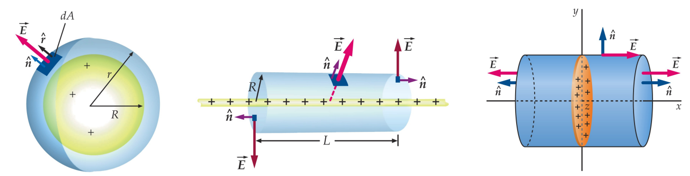
$$(1 + 2 + 5)\frac{\text{C}}{\epsilon_0} = \frac{2(q_1 + q_2 + q_3)}{\epsilon_0} = \frac{2Q}{\epsilon_0} \quad \Rightarrow \ Q = 4\text{C}.$$

Calculating $\vec{E}$ from Gauss's Law: Strategy

Design the Gaussian surface such that it reflects the symmetry of the problem at hand.

- **Use concentric Gaussian spheres in problems with spherically symmetric charge distributions.**
  The electric field is perpendicular to the Gaussian sphere ($\vec{E} \parallel d\vec{A}$).
- **Use coaxial Gaussian cylinders in problems with cylindrically symmetric charge distributions.**
  The electric field is perpendicular to the curved surface ($\vec{E} \parallel d\vec{A}$) and parallel to the flat surfaces ($\vec{E} \perp d\vec{A}$).
- **Use Gaussian cylinders with axis perpendicular to planar charge distributions.**
  The electric field is parallel to the curved surface ($\vec{E} \perp d\vec{A}$) and perpendicular to the flat surfaces ($\vec{E} \parallel d\vec{A}$).

Since the magnitude of the electric field $\vec{E}$ is constant along both curved surfaces, the integral $\oint \vec{E} \cdot d\vec{A}$ reduces to $\pm EA$, where $A = 4\pi r^2$ (sphere) or $A = 2\pi RL$ (cylinder).



tsl51

We now wish to employ Gauss's law for the calculation of the electric field generated by electrically charged, extended objects. We have pursued the same goal in the previous lecture, by using Coulomb's law.

The use of Gauss's law for the same purpose makes the calculation much simpler if certain symmetry conditions are satisfied.

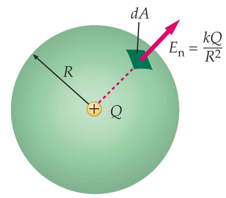
We shall consider charge distribution with spherical, cylindrical, and planar symmetry.

The key for the practicality of this method is that we choose Gaussian surface in a shape that matches the symmetry of the given charge distribution and to position the Gaussian surface in such a way that electric flux can be determined as a function of the (unknown) electric field with no need of integration.

The itemized list on the slide explains and graphically illustrates what the smart choices are for the three kinds of symmetry. The power of this method will become clear as we move to specific applications.

Calculating $\vec{E}$ from Gauss's Law: Point Charge

- Consider a positive point charge Q.
- Use a Gaussian sphere of radius R centered at the location of Q.
- Surface area of sphere: $A = 4\pi R^2$.
- Electric flux through Gaussian surface: $\Phi_E = \oint \vec{E} \cdot d\vec{A} = E(4\pi R^2)$.
- Net charge inside Gaussian surface: $Q_{in} = Q$.
- Gauss's law $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}$ becomes $E(4\pi R^2) = \frac{Q}{\epsilon_0}$.
- Electric field at radius R: $E = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} = \frac{kQ}{R^2}$.



tsl52

We have mentioned before that Gauss's law is more general than Coulomb's law. That means we can derive Gauss's law from Coulomb's law for all situations where the latter is valid. We have done that for one case on the first page of this lecture.

We cannot derive Coulomb's law from Gauss's law for all situations, only for special situations pertaining to electrostatics. A point charge Q at rest represents a situation that qualifies.

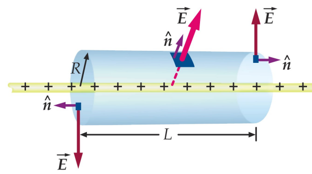
The electric field of a point charge Q is radial. We have a configuration with spherical symmetry. We put a Gaussian sphere of radius R around the point charge such that the charge is at the center.

The electric field $\vec{E}$ then has the same magnitude at all points on the sphere and its direction is the same as that of the area vector $d\vec{A}$ of an infinitesimal tile anywhere on the surface. We are thus justified to calculate the electric flux through the Gaussian sphere as the product of the area of the sphere and the (unknown) electric field at radius R.

Gauss's law then relates that flux to the charge inside, which we know to be the point charge Q that generates the electric field E. Solving that relation for the electric field thus recovers Coulomb's law from Gauss's law.

Calculating $\vec{E}$ from Gauss's Law: Charged Wire

- Consider a uniformly charged wire of infinite length.
- Charge per unit length on wire: λ (here assumed positive).
- Use a coaxial Gaussian cylinder of radius R and length L.
- Electric flux through Gaussian surface: $\Phi_E = \oint \vec{E} \cdot d\vec{A} = E(2\pi RL)$.
- Net charge charge inside Gaussian surface: $Q_{in} = \lambda L$.
- Gauss's law $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}$ becomes $E(2\pi RL) = \frac{\lambda L}{\epsilon_0}$.
- Electric field at radius R: $E = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{R}$.



tsl53

Here we calculate the electric field generated by a very long, uniformly charged wire. We have done that in the previous lecture, using Coulomb's law. Here we use Gauss's law instead.

We have a configuration with cylindrical symmetry. The (unknown) electric field points radially outward from the positively charged wire with given charge per unit length λ.

A Gaussian surface that matches the symmetry is a can of length L and radius R positioned such that its axis coincides with a stretch of wire as shown. The can has two flat surfaces and a curved surface.

There is no electric flux through the two flat surfaces. The electric field is tangential to those surfaces, thus perpendicular to the local area vectors. This produces vanishing dot products $\vec{E} \cdot d\vec{A}$.

The magnitude of the electric field is the same at all points on the curved surface of the can. Its direction is the same as the local area vector. Therefore, the electric flux is the product of the (unknown) field E and the area A of the curved surface.

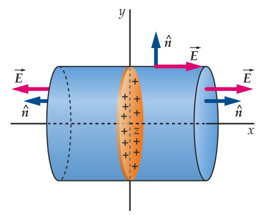
The curved surface can be unbent into rectangle with sides L and $2\pi R$, the length and the circumference of the can, respectively.

The charge inside the can is readily identified. Gauss's law then relates the charge inside the can to the flux through the can. It produces an equation that we solve for the unknown E.

The result agrees with that previously found. With the method employed here the calculation is much simpler.

Calculating $\vec{E}$ from Gauss's Law: Charged Plane Sheet

- Consider a uniformly charged plane sheet.
- Charge per unit area on sheet: σ (here assumed positive).
- Use Gaussian cylinder with cross-sectional area A placed as shown.
- Electric flux through Gaussian surface: $\Phi_E = \oint \vec{E} \cdot d\vec{A} = 2EA$.
  Net charge charge inside Gaussian surface: $Q_{in} = \sigma A$.
- Gauss's law $\oint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}$ becomes $2EA = \frac{\sigma A}{\epsilon_0}$.
- Electric field at both ends of cylinder: $E = \frac{\sigma}{2\epsilon_0} = 2\pi k\sigma$ (pointing away from sheet).
- Note that E does not depend on the distance from the sheet.



tsl54

We can use the can from the previous page as the Gaussian surface for the calculation of the electric field generated on both sides of a large, plane, uniformly charged sheet. We know the result already from the previous lecture, where we used a method based on Coulomb's law.

Here again, reproducing the result takes much less effort when we employ Gauss's law. Symmetry dictates that the field is pointing away from a positively charged sheet with uniform charge per unit area σ. Symmetry does not require the field to be uniform. Therefore, we must allow the possibility that the field strength depends on distance from the sheet.

When we position the can symmetrically across the sheet as shown, then there is zero electric flux through its curved surface. The two flat surfaces have equal electric flux, being equidistant from the sheet.

The flux through the can is equal to the (unknown) electric field E at the distance of the flat surfaces from the sheet, multiplied by the area $2A$ of the two flat surfaces.

The charge inside the can is proportional to the area of the sheet enclosed by the can, which is A again.

The next step relates flux through the can with charge inside the can as Gauss's law demands. That relation solved for the unknown E reproduces the familiar result, which is independent of the distance from the sheet, thus confirming that the electric field is indeed uniform.

Note also that the can in this application, unlike in the previous one, does not have to be cylindrical in shape. Important is that it has two parallel flat surfaces of any shape joined by an orthogonal surface at their perimeters.

Also, neither in this nor in the previous application did the length of the can matter. In the next application it will matter.

Calculating $\vec{E}$ from Gauss's Law: Charged Slab

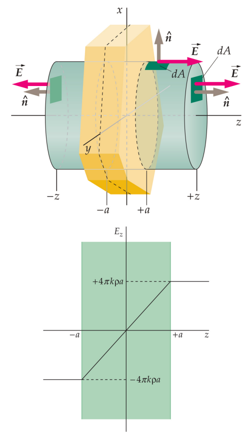
- Consider a uniformly charged slab.
- Charge per unit volume on slab: ρ.
- Use Gaussian cylinder as shown.

- Total electric flux: $\Phi_E = 2|E_z|A$.

- Net charge inside: $Q_{in} = \begin{cases} 2\rho A|z| & (|z| \leq a) \\ 2\rho Aa & (|z| \geq a) \end{cases}$

- Gauss's law: $2|E_z|A = \begin{cases} \frac{2\rho A|z|}{\epsilon_0} & (|z| \leq a) \\ \frac{2\rho Aa}{\epsilon_0} & (|z| \geq a) \end{cases}$

- Electric field: $E_z = \begin{cases} -\frac{\rho a}{\epsilon_0} & (z \leq -a) \\ \frac{\rho z}{\epsilon_0} & (-a \leq z \leq a) \\ \frac{\rho a}{\epsilon_0} & (z \geq a) \end{cases}$



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Here we continue with the same technique to analyze the case of a large slab of width $2a$. The uniform charge per unit volume ρ is assumed to be positive. The can from the previous page is once again a convenient shape of Gaussian surface.

The slab is positioned in the xy plane. Hence the electric field only has a z-component and is directed away from the slab. We allow its strength to be dependent on the distance from the xy-plane. The electric flux, therefore, is as stated in the fourth item on the slide.

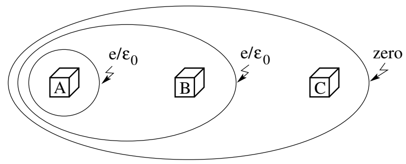
For the analysis via Gauss's law, we must distinguish two cases:

- If the length of the can is larger than the width of the slab, we are determining the electric field at positions $\pm z$ outside the slab. In this case the charge inside the can is independent of the positions of the flat surfaces. The application of Gauss's law then predicts an electric field that is uniform again, independent of the distance from the slab.
- If the length of the can is smaller than the width of the slab, we are determining the electric field at positions $\pm z$ inside the slab. In this case the charge does depend the positions of the flat surfaces. The application of Gauss's law then predicts an electric field that varies linearly with distance from the center of the slab.

The function E_z is plotted on the lower right. Negative (positive) values mean a field directed left (right).

Gaussian surface problem (3)

A proton, a neutron, and an electron are placed in different boxes. The electric fluxes through the three Gaussian surfaces are as indicated, where e stands for the elementary charge.



Name the particle in each box.

tsl47

This is the quiz for lecture 5.

We return to unknown charged particles inside Gaussian surfaces with known electric flux through them. In this instance, we have three boxes, each containing an elementary particle, either a proton with charge $+e$, an electron with charge $-e$, or a neutron with no electric charge.

Which particle is in which box?