Dynamics of Charged Particle:

- Newton's equation of motion: $\vec{F} = m\vec{a}$.
- Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$.

Dynamics of Electric and Magnetic Fields:

- Gauss' law for electric field: $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$.
- Gauss' law for magnetic field: $\oint \vec{B} \cdot d\vec{A} = 0$.
- Faraday's law: $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$, where $\Phi_B = \int \vec{B} \cdot d\vec{A}$.
- Ampère's law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$, where $\Phi_E = \int \vec{E} \cdot d\vec{A}$.

Maxwell's equations: 4 relations between fields $(\vec{E}, \vec{B})$ and sources (q, I).

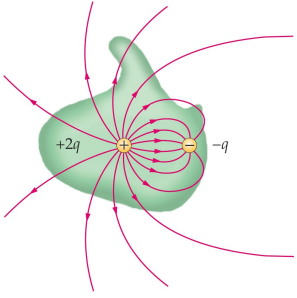
Gauss's Law for Electric Field

The net electric flux Φ_E through any closed surface is equal to the net charge Q_{in} inside divided by the permittivity constant ϵ_0:

$$\oint \vec{E} \cdot d\vec{A} = 4\pi k Q_{in} = \frac{Q_{in}}{\epsilon_0} \qquad \text{i.e.} \qquad \Phi_E = \frac{Q_{in}}{\epsilon_0} \qquad \text{with} \quad \epsilon_0 = 8.854 \times 10^{-12} \mathrm{C^2 N^{-1} m^{-2}}$$

The closed surface can be real or fictitious. It is called "Gaussian surface".
The symbol $\oint$ denotes an integral over a closed surface in this context.

- Gauss's law is a general relation between electric charge and electric field.
- In electrostatics: Gauss's law is equivalent to Coulomb's law.
- Gauss's law is one of four Maxwell's equations that govern cause and effect in electricity and magnetism.



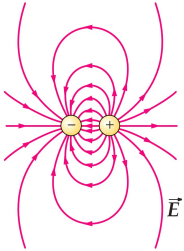
Gauss's Law for Magnetic Field

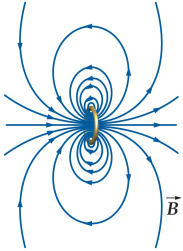
The net magnetic flux Φ_B through any closed surface is equal to zero:

$$\oint \vec{B} \cdot d\vec{A} = 0.$$

There are no magnetic charges. Magnetic field lines always close in themselves. No matter how the (closed) Gaussian surface is chosen, the net magnetic flux through it always vanishes.

The figures below illustrate Gauss's laws for the electric and magnetic fields in the context of an electric dipole (left) and a magnetic dipole (right).

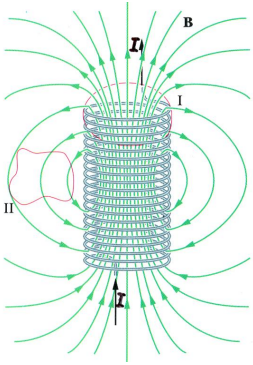


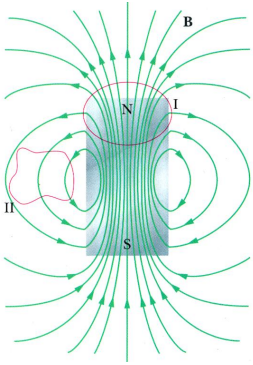


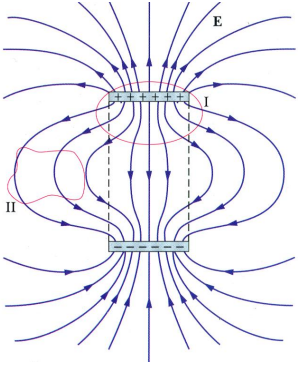
Gauss' Law for Electric and Magnetic Fields

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\epsilon_0}$$

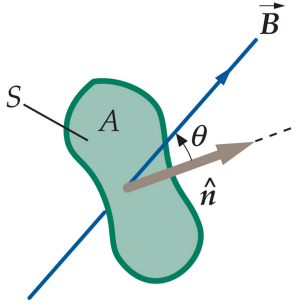






Magnetic flux and Faraday's law

- Magnetic field $\vec{B}$ (given)
- Surface S with perimeter loop (given)
- Surface area A (given)
- Area vector $\vec{A} = A\hat{n}$ (my choice)
- Positive direction around perimeter: ccw (consequence of my choice)
- Magnetic flux: $\Phi_B = \int \vec{B} \cdot d\vec{A} = \int \vec{B} \cdot \hat{n} dA$
- Consider situation with $\frac{d\vec{B}}{dt} \neq 0$
- Induced electric field: $\vec{E}$
- Induced EMF: $\mathcal{E} = \oint \vec{E} \cdot d\vec{\ell}$ (integral ccw around perimeter)
- Faraday's law: $\mathcal{E} = -\frac{d\Phi_B}{dt}$



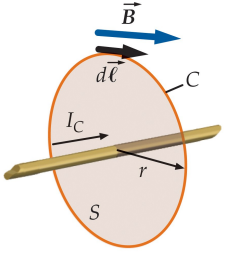
Ampère's Law (Restricted Version)

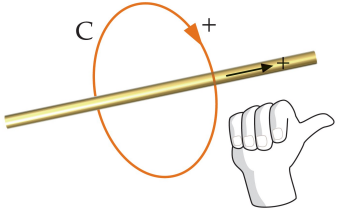
The circulation integral of the magnetic field $\vec{B}$ around any closed curve (loop) C is equal to the net electric current I_C flowing through the loop:

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_C, \qquad \text{with} \quad \mu_0 = 4\pi \times 10^{-7} \text{Tm/A}$$

The symbol $\oint$ denotes an integral over a closed curve in this context.
Note: Only the component of $\vec{B}$ tangential to the loop contributes to the integral.

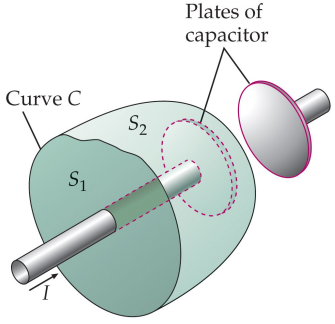
The positive current direction through the loop is determined by the right-hand rule.

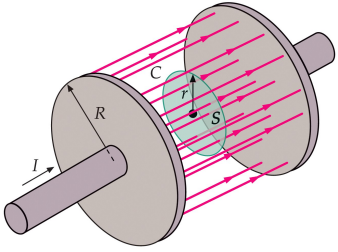




Ampère's law (Full Version)

- Conduction current: I.
- Displacement current: $I_D = \epsilon_0 \dfrac{d\Phi_E}{dt}$.
- Ampère's law: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0(I + I_D) = \mu_0 I + \mu_0 \epsilon_0 \dfrac{d\Phi_E}{dt}$.

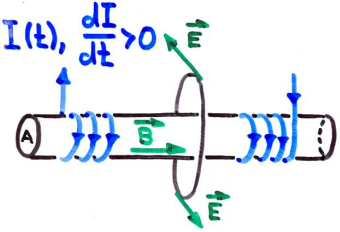


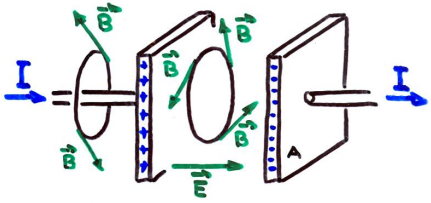


Faraday's law and Ampère's law

$$\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$





Traveling Waves

Mechanical waves travel in some medium.
Examples: sound wave, violin string, surface water wave.
While the wave propagates, the medium undergoes periodic motion.

Distinguish:

(1) direction of wave propagation,
(2) direction in which medium moves.

Transverse wave: (1) and (2) are **perpendicular** to each other.

Longitudinal wave: (1) and (2) are **parallel** to each other.

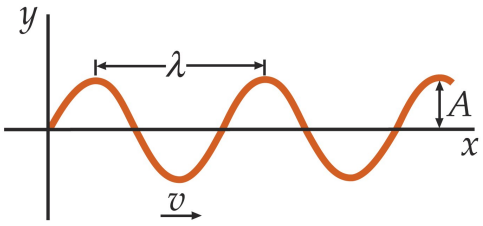
Electromagnetic waves are transversely oscillating electric and magnetic fields.
Electromagnetic waves travel in the vacuum. There is no medium.

Waves transport energy and, in some cases, information, but not the medium itself (if there is a medium).

Sinusoidal Transverse Traveling Wave

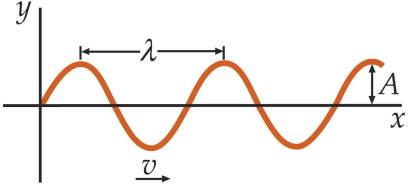
Wave function: $y(x,t) = A\sin(kx - \omega t)$

- $k = \dfrac{2\pi}{\lambda}$ (wave number)
- λ (wavelength)
- $\omega = \dfrac{2\pi}{T} = 2\pi f$ (angular frequency)
- $f = \dfrac{\omega}{2\pi} = \dfrac{1}{T}$ (frequency)
- T (period)
- $c = \dfrac{\lambda}{T} = \lambda f = \dfrac{\omega}{k}$ (wave speed)



Wave Equation

- $y(x,t) = A\sin(kx - \omega t)$ (displacement)
- $v(x,t) = \dfrac{\partial y}{\partial t} = -\omega A\cos(kx - \omega t)$ (velocity)
- $a(x,t) = \dfrac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx - \omega t)$ (acceleration)
- $\dfrac{\partial y}{\partial x} = kA\cos(kx - \omega t)$ (slope of wave form)
- $\dfrac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx - \omega t)$ (curvature of wave form)
- $\dfrac{\partial^2 y/\partial t^2}{\partial^2 y/\partial x^2} = \dfrac{\omega^2}{k^2} = c^2$ (ratio of second derivatives)
- Wave equation: $\dfrac{\partial^2 y}{\partial t^2} = c^2\,\dfrac{\partial^2 y}{\partial x^2}$



Electromagnetic Plane Wave (1)

Maxwell's equations for electric and magnetic fields in free space (no sources):

- Gauss' laws: $\oint \vec{E} \cdot d\vec{A} = 0, \qquad \oint \vec{B} \cdot d\vec{A} = 0.$
- Faraday's and Ampère's laws: $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}, \qquad \oint \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$

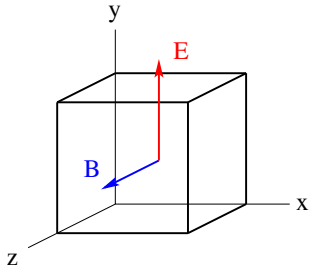
Consider fields of particular directions and dependence on space:

$$\vec{E} = E_y(x,t)\hat{j}, \quad \vec{B} = B_z(x,t)\hat{k}.$$

Gauss' laws are then automatically satisfied.

Use the cubic Gaussian surface to show that

- the net electric flux Φ_E is zero,
- the net magnetic flux Φ_B is zero.



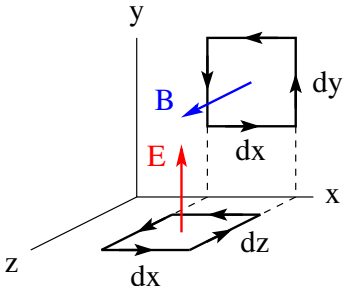
Electromagnetic Plane Wave (2)

- Faraday's law, $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$,
  applied to loop in (x,y)-plane becomes
  $[E_y(x+dx,t) - E_y(x,t)]dy = -\frac{\partial}{\partial t} B_z(x,t) dxdy$

  $\Rightarrow \quad \frac{\partial}{\partial x} E_y(x,t) = -\frac{\partial}{\partial t} B_z(x,t) \qquad (\mathrm{F})$

- Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$,
  applied to loop in (x,z)-plane becomes
  $[-B_z(x+dx,t) + B_z(x,t)]dz = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_y(x,t) dxdz$

  $\Rightarrow \quad -\frac{\partial}{\partial x} B_z(x,t) = \mu_0 \epsilon_0 \frac{\partial}{\partial t} E_y(x,t) \qquad (\mathrm{A})$



Electromagnetic Plane Wave (3)

Take partial derivatives $\frac{\partial}{\partial x}$(F) and $\frac{\partial}{\partial t}$(A): $\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial t \partial x}$, $-\frac{\partial^2 B_z}{\partial t \partial x} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$.

$\Rightarrow \frac{\partial^2 E_y}{\partial t^2} = c^2 \frac{\partial^2 E_y}{\partial x^2}$ (E) (wave equation for electric field).

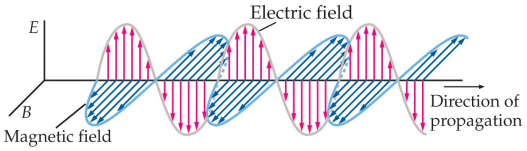
Take partial derivatives $\frac{\partial}{\partial t}$(F) and $\frac{\partial}{\partial x}$(A): $\frac{\partial^2 E_y}{\partial t \partial x} = -\frac{\partial^2 B_z}{\partial t^2}$, $-\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t \partial x}$.

$\Rightarrow \frac{\partial^2 B_z}{\partial t^2} = c^2 \frac{\partial^2 B_z}{\partial x^2}$ (B) (wave equation for magnetic field).

$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ (speed of light).

Sinusoidal solution:

- $E_y(x,t) = E_{max} \sin(kx - \omega t)$
- $B_z(x,t) = B_{max} \sin(kx - \omega t)$



Electromagnetic Plane Wave (4)

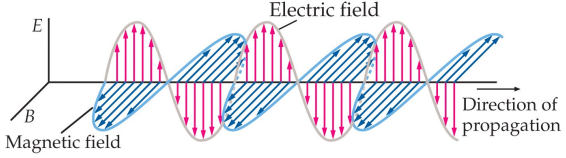
For given wave number k the angular frequency ω is determined, for example by substitution of $E_{max}\sin(kx-\omega t)$ into (E).

For given amplitude E_{max} the amplitude B_{max} is determined, for example, by substituting $E_{max}\sin(kx-\omega t)$ and $B_{max}\sin(kx-\omega t)$ into (A) or (F).

$$\Rightarrow \quad \frac{\omega}{k} = \frac{E_{max}}{B_{max}} = c.$$

The direction of wave propagation is determind by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\,\vec{E}\times\vec{B}.$$



Energy Transport in Electromagnetic Plane Wave

Fields: $E_y(x,t) = E_{max}\sin(kx-\omega t), \quad B_z(x,t) = B_{max}\sin(kx-\omega t).$

Energy density: $u(x,t) = \frac{1}{2}\epsilon_0 E_y^2(x,t) + \frac{1}{2\mu_0}B_z^2(x,t).$ [J/m^3]

Use the amplitude relations $\epsilon_0 E_{max}^2 = \epsilon_0 c^2 B_{max}^2 = \frac{1}{\mu_0}B_{max}^2.$

$$u(x,t) = \epsilon_0 E_{max}^2 \sin^2(kx-\omega t) = \frac{1}{\mu_0}B_{max}^2\sin^2(kx-\omega t) = \frac{E_{max}B_{max}}{c\mu_0}\sin^2(kx-\omega t).$$

Energy transported across area A in time dt: $dU(x,t) = u(x,t)Acdt.$ [J]

Power transported per unit area: $\frac{1}{A}\frac{dU}{dt} = u(x,t)c = S(x,t).$ [W/m^2]

Intensity (average power transported per unit area):

$$I = \bar{S} = \frac{E_{max}B_{max}}{2\mu_0} = \frac{\epsilon_0 c}{2}E_{max}^2 = \frac{c}{2\mu_0}B_{max}^2. \quad [\text{W/m}^2]$$

Momentum Transport in Electromagnetic Plane Wave

The momentum transported by an electromagnetic wave is proportional to the energy transported.

Momentum density: $\frac{\vec{p}}{V} = \frac{\vec{S}}{c^2}$, where $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$ is the Poynting vector.

When the wave is absorbed by a material surface it exerts an impulse $\vec{F}dt = \Delta\vec{p}$.

The resulting radiation pressure is the average force per unit area:

$$P_{abs} = \frac{\bar{F}}{A} = \frac{p}{Adt} = \frac{p}{Adx}\frac{dx}{dt} = \frac{p}{V}c = \frac{\bar{S}}{c} = \frac{I}{c}.$$

The radiation pressure exerted by a reflected wave is twice as large: $P_{ref} = \frac{2\bar{S}}{c} = \frac{2I}{c}$.