Conservation Laws and Symmetry

Consider an isolated system described by generalized coordinates in an inertial reference frame: $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. The following conservation laws can be derived from general properties of space and time.

- Homogeneity of time leads to conservation of energy.

$$\frac{\partial L}{\partial t} = 0 \;\Rightarrow\; L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n).$$

$$\frac{dL}{dt} = \sum_j \left[\frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right] = \sum_j \left[\dot{q}_j \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} + \ddot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right] = \sum_j \frac{d}{dt} \left[\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right].$$

$$\frac{d}{dt} \left[L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right] = 0 \;\Rightarrow\; \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = E(q_j, \dot{q}_j) = \text{const.}$$

$$L(q_j, \dot{q}_j) = T(q_j, \dot{q}_j) - V(q_j), \quad \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = 2T \quad [\text{mex155}]$$

$$E(q_j, \dot{q}_j) = T(q_j, \dot{q}_j) + V(q_j).$$

A generalized coordinate q_l which does not appear in the Lagrangian $L(q_j, \dot{q}_j)$ is called *cyclic*. The generalized momentum p_l conjugate to a cyclic coordinate is conserved:

$$\frac{\partial L}{\partial q_l} = 0 \;\Rightarrow\; \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_l} = 0 \;\Rightarrow\; \frac{\partial L}{\partial \dot{q}_l} \doteq p_l = \text{const.}$$

- Homogeneity of space leads to conservation of linear momentum.

  The Lagrangian is invariant under global translations.
  Therefore, the center-of-mass coordinates are cyclic.
  Therefore, the total linear momentum vector is conserved.

- Isotropy of space leads to conservation of angular momentum.

  The Lagrangian is invariant under global rotations.
  Therefore, the angle of a global rotation about any axis is cyclic.
  Therefore, the total angular momentum vector is conserved.