Motion in time on elliptic Kepler orbit [mln19]

Use orbital equation and specifications from [msl23]:

$$\frac{p}{r} = 1 + e\cos\vartheta, \quad a = \frac{\kappa}{2|E|}, \quad e = \sqrt{1 - \frac{2|E|\ell^2}{m\kappa^2}}, \quad p = a(1-e^2).$$

Use the formal solution with $E < 0$, $V(r) = -\kappa/r$, $\kappa = GmM$ from [mln18]:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left[E + \frac{\kappa}{r} - \frac{\ell^2}{2mr^2}\right]}} = \sqrt{\frac{m}{2|E|}} \int \frac{r\,dr}{\sqrt{-r^2 + \frac{\kappa}{|E|}\,r - \frac{\ell^2}{2m|E|}}}.$$

$$\Rightarrow \; t = \sqrt{\frac{ma}{\kappa}} \int \frac{r\,dr}{\sqrt{a^2e^2 - (a-r)^2}}.$$

Cartesian coordinates: $x = r\cos\vartheta = (p-r)/e,\; y = \sqrt{r^2 - x^2}$.

Angular coordinate: $\cos\vartheta = (p-r)/er$.

Use identity: $\cos\vartheta = \dfrac{1-\tan^2(\vartheta/2)}{1+\tan^2(\vartheta/2)}$.

Parametrization: $r(\psi) = a(1 - e\cos\psi) \quad : \; 0 \leq \psi \leq 2\pi$.

Parametric representation for the motion in time:

$$r(\psi) = a(1 - e\cos\psi),$$

$$\tan\frac{\vartheta(\psi)}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2},$$

$$x(\psi) = a(\cos\psi - e),$$

$$y(\psi) = a\sqrt{1-e^2}\sin\psi,$$

$$t = \sqrt{\frac{ma}{\kappa}} \int d\psi\, a(1 - e\cos\psi) = \sqrt{\frac{ma^3}{\kappa}}(\psi - e\sin\psi).$$

Period of motion: $\tau = 2\pi\sqrt{\dfrac{ma^3}{\kappa}}$.

Circular limit: $e = 0 \;\Rightarrow\; r = a = \text{const},\; \vartheta = \psi,\; t = \tau\,\dfrac{\vartheta}{2\pi}$.