

## Euler's Equations [mln27]

Equation of motion in inertial frame:  $\left(\frac{d\mathbf{L}}{dt}\right)_I = \mathbf{N}$ .

Equation of motion in (rotating) body frame:  $\left(\frac{d\mathbf{L}}{dt}\right)_R + \vec{\omega} \times \mathbf{L} = \mathbf{N}$ .

Proof:

$$\mathbf{N} = \frac{d}{dt} \left[ \sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha} \times (\vec{\omega} \times \mathbf{r}_{\alpha}) \right] = \frac{d}{dt} \left[ \sum_i L_i \mathbf{e}_i \right] = \dot{L}_i \mathbf{e}_i + L_i \dot{\mathbf{e}}_i.$$

$$\text{Use } L_i = \sum_{ij} I_{ij} \omega_j. \quad \dot{\mathbf{e}}_i = \vec{\omega} \times \mathbf{e}_i.$$

$$\Rightarrow \mathbf{N} = \sum_{ij} I_{ij} \dot{\omega}_j \mathbf{e}_i + \vec{\omega} \times \sum_{ij} I_{ij} \omega_j \mathbf{e}_i.$$

Choose body frame with principal coordinate axes:  $L_i = I_i \omega_i$ ,  $i = 1, 2, 3$ .

Euler's equations:

$$\begin{aligned} I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) &= N_1 \\ I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) &= N_2 \\ I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) &= N_3 \end{aligned}$$

A purely rotating rigid body has 3 degrees of freedom.

The associated Lagrange equations are three 2<sup>nd</sup> order ODEs.

The solution via Euler's equations proceeds in two steps:

1. Euler's equations themselves are three 1<sup>st</sup> order ODEs for  $\omega_1, \omega_2, \omega_3$ .
2. The transformation to the inertial frame,  $\omega_i = \omega_i(\phi, \theta, \psi; \dot{\phi}, \dot{\theta}, \dot{\psi})$  amounts to solving another three 1<sup>st</sup> order ODEs.