Routhian Function

Goal: systematic elimination of cyclic coordinates in the Lagrangian formulation of mechanics.

System with \( n \) generalized coordinates.

Lagrangian: \( L(q_{k+1}, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \Rightarrow q_1, \ldots, q_k \) are cyclic.

Routhian: \( R(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t) = L - \sum_{i=1}^{k} \beta_i \dot{q}_i \).

The relations \( \beta_i = \frac{\partial L}{\partial \dot{q}_i} = \text{const} \), \( i = 1, \ldots, k \) are to be inverted into \( \dot{q}_i = \dot{q}_i(q_{k+1}, \ldots, q_n, \dot{q}_{k+1}, \ldots, \dot{q}_n, \beta_1, \ldots, \beta_k, t), \) \( i = 1, \ldots, k \).

Compare coefficients of the variations

\[
\delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i} \delta \beta_i + \frac{\partial R}{\partial t} \delta t,
\]

\[
\delta \left( L - \sum_{i=1}^{k} \beta_i \dot{q}_i \right) = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \dot{L}}{\partial q_i} \delta \dot{q}_i - \sum_{i=1}^{k} \beta_i \delta \dot{q}_i - \sum_{i=1}^{k} \dot{q}_i \delta \beta_i + \frac{\partial L}{\partial t} \delta t.
\]

Resulting relations between partial derivatives:

\[ \frac{\partial R}{\partial q_i} = \frac{\partial L}{\partial q_i}, \quad \frac{\partial R}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}, \quad i = k + 1, \ldots, n, \]

\[ \frac{\partial R}{\partial t} = \frac{\partial L}{\partial t}; \quad \dot{q}_i = -\frac{\partial R}{\partial \beta_i}, \quad i = 1, \ldots, k. \]

Lagrange equations for the noncyclic coordinates:

\[ \frac{\partial R}{\partial q_i} - \frac{d}{dt} \frac{\partial R}{\partial \dot{q}_i} = 0, \quad i = k + 1, \ldots, n. \]

Time evolution of cyclic coordinates:

\[ q_i(t) = -\int dt \frac{\partial R}{\partial \beta_i}, \quad i = 1, \ldots, k. \]