Relativistic Energy II [mln65]

Relativistic adaptation of Newton's equation of motion:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \qquad \mathbf{p} = \frac{m_0\mathbf{v}}{\sqrt{1-v^2/c^2}}.$$

Conservative force: $\mathbf{F} = -\nabla U$.

Work and potential energy: $W_{12} = \int_1^2 d\mathbf{r}\cdot\mathbf{F} = -(U_2 - U_1)$.

Work and relativistic energy:

$$W_{12} = \int_1^2 dt\,\mathbf{v}\cdot\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) = \int_1^2 dt\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right) = E_2 - E_1,$$

where we have used,

$$\begin{aligned}\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) &= \frac{m_0(d\mathbf{v}/dt)}{\sqrt{1-v^2/c^2}} - \frac{m_0\mathbf{v}}{2(1-v^2/c^2)^{3/2}}\cdot\left(-\frac{2\mathbf{v}}{c^2}\right)\frac{d\mathbf{v}}{dt} \\ &= \frac{m_0(d\mathbf{v}/dt)}{\sqrt{1-v^2/c^2}}\left(1+\frac{v^2/c^2}{1-v^2/c^2}\right) = \frac{m_0(d\mathbf{v}/dt)}{(1-v^2/c^2)^{3/2}},\end{aligned}$$

$$\frac{d}{dt}\left(\frac{m_0c^2}{\sqrt{1-v^2/c^2}}\right) = -\frac{m_0c^2}{2(1-v^2/c^2)^{3/2}}\left(-\frac{2\mathbf{v}}{c^2}\right)\cdot\frac{d\mathbf{v}}{dt} = \mathbf{v}\cdot\frac{m_0(d\mathbf{v}/dt)}{(1-v^2/c^2)^{3/2}}.$$

Energy conservation: $E_1 + U_1 = E_2 + U_2$.

Work and nonrelativistic energy (for comparison):

$$W_{12} = \int_1^2 dt\,\mathbf{v}\cdot\frac{d}{dt}m_0\mathbf{v} = \int_1^2 dt\frac{d}{dt}\left(\frac{1}{2}m_0v^2\right) = T_2 - T_1.$$

Space-time four-vector: $x_\mu \doteq (ct, x_1, x_2, x_3)$.

Energy-momentum four-vector: $p_\mu \doteq (E/c, p_1, p_2, p_3)$.

Lorentz transformation:

$$x_1' = \frac{x_1 - vt}{\sqrt{1 - v^2/c^2}}, \quad x_2' = x_2, \quad x_3' = x_3, \quad t' = \frac{t - (v/c^2)x_1}{\sqrt{1 - v^2/c^2}}.$$

$$p_1' = \frac{p_1 - (v/c^2)E}{\sqrt{1 - v^2/c^2}}, \quad p_2' = p_2, \quad p_3' = p_3, \quad E' = \frac{E - vp_1}{\sqrt{1 - v^2/c^2}}.$$

Invariant quantity:[1] $E^2 - \mathbf{p}^2c^2 = m_0^2c^4$.

Relativistic energy-momentum relation: $E = \sqrt{\mathbf{p}^2c^2 + m_0^2c^4}$.

- Nonrelativistic limit: $E \simeq m_0c^2 + \dfrac{\mathbf{p}^2}{2m_0}$.
- Ultrarelativistic limit: $E \simeq \mathbf{p}c$.

Lorentz transformation of radiant energy: Set $p_1 = E/C$ and $p_1' = E'/c$.

$$\Rightarrow \frac{E'}{c} = \frac{E/c - (v/c)(E/c)}{\sqrt{1 - v^2/c^2}} = \frac{E}{c}\sqrt{\frac{1 - v/c}{1 + v/c}}.$$

This result echoes the Doppler shift. It is of importance in quantum mechanics. The energy of photons is $E = h\nu$. In the moving frame, photons still travel at the speed of light.

Massive particle lose energy and momentum in the moving frame by going more slowly. Photons do it via redshift without slowing down.

[1] For a quick verification set $p_1 = p_2 = p_3 = 0$ and calculate $E'^2 - (p_1')^2$.