Central Force Motion: One-Body Problem [mln67]

Reduction to one degree of freedom:

Consider a particle of mass m moving in a central potential:

Lagrangian: $L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 - V(|\mathbf{r}|)$.

Conservation of angular momentum: $\mathbf{L} = \mathbf{r} \times m\dot{\mathbf{r}} = \text{const.}$

- Case $\mathbf{L} = 0$: One degree of freedom.
  - Purely radial motion: $\mathbf{r} \parallel \dot{\mathbf{r}} \quad \Rightarrow \; L(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 - V(r)$.
  - Energy conservation: $E(r, \dot{r}) = \frac{1}{2} m \dot{r}^2 + V(r)$.
  - Reduction to quadrature (see [mln4]).

- Case $\mathbf{L} \neq 0$: Two separable degrees of freedom.
  - Motion in plane perpendicular to $\mathbf{L}$.
  - Transformation to polar coordinates: $x = r\cos\vartheta, \; y = r\sin\vartheta$.
  - Lagrangian: $L(r, \dot{r}, \dot{\vartheta}) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\vartheta}^2) - V(r)$.
  - Cyclic coordinate: ϑ.
  - Conserved angular momentum: $\ell = \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta} = \text{const.}$
  - Routhian: $R(r, \dot{r}; \ell) = L - \ell\dot{\vartheta} = \frac{1}{2} m\dot{r}^2 - \frac{\ell^2}{2mr^2} - V(r)$.
  - Effective potential for radial motion: $\tilde{V}(r; \ell) \doteq V(r) + \frac{\ell^2}{2mr^2}$.
  - Conserved energy: $E(r, \dot{r}; \ell) = \frac{1}{2} m\dot{r}^2 + \tilde{V}(r; \ell)$.
  - Reduction to quadrature (see [mln4]).
  - Integral for angular motion: $\vartheta(t) = \vartheta_0 + \frac{\ell}{m} \int_0^t \frac{dt}{mr^2(t)}$.