D'Alembert's Principle [mln7]

Consider virtual displacements $\delta\mathbf{r}_i$:

- they are infinitesimal;
- they satisfy the equations of constraint;
- they are instantaneous ($\delta t = 0$).

For scleronomic constraints, the paths of real and virtual displacements are the same; for rheonomic constraints, they differ.

Newton's equations of motion: $m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{Z}_i, \quad i = 1, \ldots, N.$

Generalized coordinates: $q_1, \ldots, q_{3N-k}$.

Cartesian coordinates: $\mathbf{r}_i(q_1, \ldots, q_{3N-k}, t), \quad i = 1, \ldots, N.$

Virtual displacements: $\delta\mathbf{r}_i = \displaystyle\sum_{j=1}^{3N-k} \frac{\partial\mathbf{r}_i}{\partial q_j}\,\delta q_j.$

D'Alembert's principle: $\sum_{i=1}^{N} \mathbf{Z}_i \cdot \delta\mathbf{r}_i = 0.$
The forces of constraint perform zero net work.

A consequence of D'Alembert's principle is
D'Alembert's equation: $\sum_{i=1}^{N} (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \delta\mathbf{r}_i = 0.$
It does no longer contain the forces of constraint.

Transformation to independent (generalized) coordinates,

$$\sum_{j=1}^{3N-k} \left[\sum_{i=1}^{N} (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} \right] \delta q_j = 0,$$

results in $3N - k$ equations of motion, one for each remaining degree of freedom:

$$\sum_{i=1}^{N} (m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \frac{\partial\mathbf{r}_i}{\partial q_j} = 0, \quad j = 1, \ldots, 3N - k.$$

Applications:

- Plane pendulum II [mex134]
- Heavy particle sliding inside cone II [mex135]