Classification of Fixed Points in Plane [mln73]

Equation of motion: $m\ddot{x} = F(x, \dot{x}) \quad \Rightarrow \ \dot{x} = y, \ \dot{y} = F(x, y)/m$.

Velocity vector field: $\mathbf{v}(\mathbf{r}) = \mathbf{v}(x, y) = (\dot{x}, \dot{y}) = (v_x, v_y)$.

Fixed point: $\mathbf{v}(\mathbf{r}_k) = 0 \quad \Rightarrow \ (\dot{x}, \dot{y}) = 0$ at $(x, y) = (x_k, y_k)$.

Linearized velocity field around fixed point $\mathbf{r}_k$:

$$\mathbf{v} = \mathbf{A} \cdot (\mathbf{r} - \mathbf{r}_k) + \mathrm{O}(\mathbf{r} - \mathbf{r}_k)^2$$

with Jacobian matrix

$$\mathbf{A} = \begin{pmatrix} \partial v_x/\partial x & \partial v_x/\partial y \\ \partial v_y/\partial x & \partial v_y/\partial y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Nature of fixed point depends on eigenvalues of $\mathbf{A}$:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \quad \Rightarrow \ \lambda^2 - \tau\lambda + \delta = 0,$$

where $\delta = ad - bc$ is the *determinant* and $\tau = a + d$ the *trace*.

Solution:

$$\lambda = \frac{\tau}{2} \pm \sqrt{\frac{\tau^2}{4} - \delta}.$$

Three types of fixed points:

- ☐ Type 1: $\tau^2 > 4\delta \quad \Rightarrow \ \lambda_1 \neq \lambda_2$, real:
  - $\delta > 0 \quad \Rightarrow$ *node* (attractor if $\tau < 0$, repellor if $\tau > 0$),
  - $\delta < 0 \quad \Rightarrow$ *hyperbolic point*.

- ☐ Type 2: $\tau^2 < 4\delta \quad \Rightarrow \ \lambda_1 = \lambda_2^*$, complex conjugate:
  - $\Re\{\lambda\} \neq 0 \quad \Rightarrow$ *spiral* (attractor if $\tau < 0$, repellor if $\tau > 0$),
  - $\Re\{\lambda\} = 0 \quad \Rightarrow$ *elliptic point*.

- ☐ Type 3: $\tau^2 = 4\delta \quad \Rightarrow \ \lambda_1 = \lambda_2$, real:
  - $b = c = 0 \quad \Rightarrow$ *star* (attractor if $\tau < 0$, repellor if $\tau > 0$),
  - $bc \neq 0 \quad \Rightarrow$ *improper node* (attractor if $\tau < 0$, repellor if $\tau > 0$).

Conservative force implies area-preserving flow.
Consequence: $\tau = 0 \quad \Rightarrow$ no repellors or attractors.
Only elliptic or hyperbolic fixed points are realized.