

Lagrange equations derived from D'Alembert's principle [mln8]

D'Alembert's equation:
$$\sum_{j=1}^{3N-k} \left[\sum_{i=1}^N (m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i) \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j = 0$$

- $\mathbf{F}_i(\mathbf{r}_j, \dot{\mathbf{r}}_j, t) \Rightarrow \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = Q_j, \quad j = 1, \dots, 3N - k.$

$$Q_j(q_1, \dots, q_{3N-k}, \dot{q}_1, \dots, \dot{q}_{3N-k}, t) \doteq \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}.$$

- $\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{i=1}^N \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} \right].$

$$\frac{d\mathbf{r}_i}{dt} = \sum_{l=1}^{3N-k} \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \mathbf{r}_i}{\partial t} = \dot{\mathbf{r}}_i(q_j, \dot{q}_j, t) \Rightarrow \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}.$$

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \right).$$

Kinetic energy: $T(q_j, \dot{q}_j, t) = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2.$

$$\Rightarrow \sum_{j=1}^{3N-k} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0 \quad \text{with independent } \delta q_j.$$

$$\Rightarrow \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1, \dots, 3N - k.$$

Assumption: $\mathbf{F}_i = -\nabla_i \tilde{V}, \quad \tilde{V}(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = V(q_1, \dots, q_{3N-k}, t).$

$$\Rightarrow Q_j = Q_j(q_1, \dots, q_{3N-k}, t) = -\frac{\partial V}{\partial q_j}, \quad \frac{\partial V}{\partial \dot{q}_j} = 0.$$

Lagrangian: $L(q_j, \dot{q}_j, t) \doteq T(q_j, \dot{q}_j, t) - V(q_j, t).$

Lagrange equations: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1, \dots, 3N - k.$

Generalized momenta: $p_j \doteq \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1, \dots, 3N - k.$