# Actions and Angles for Librations [mln93]

Hamiltonian: $H(q,p) = \frac{p^2}{2m} + V(q) = E = \text{const.}$

Canonical momentum: $p(q,E) = \pm\sqrt{2m\,[E - V(q)]}$.

Action $J$ and Hamiltonian $K(J)$ from area $A$ inside trajectory:

$$A = \oint dq\, p(q,E) = 2\int_{q_1}^{q_2} dq \sqrt{2m\,[E - V(q)]} = \int_0^{2\pi} d\theta\, J = 2\pi J.$$

$$\Rightarrow\; J(E) = \frac{1}{\pi}\int_{q_1}^{q_2} dq \sqrt{2m\,[E - V(q)]} \quad \Rightarrow\; E = K(J) = H(p,q).$$

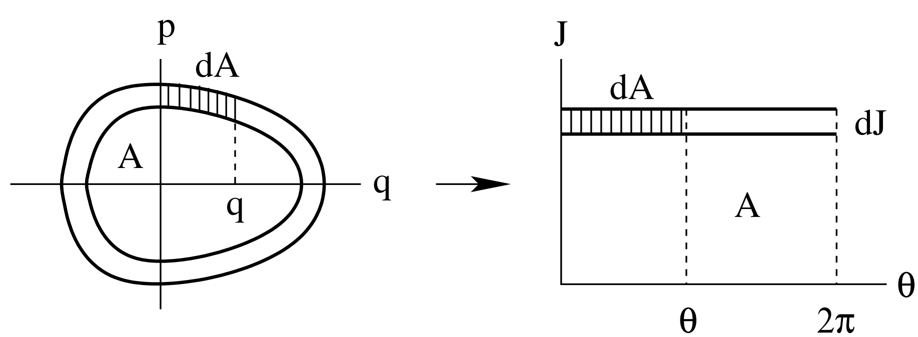
Angle variable $\theta(q,J)$ from area $dA$ between nearby trajectories:

$$dA = \int_0^q dq\,[p(q, J + dJ) - p(q,J)] = dJ \int_0^q dq\, \frac{\partial}{\partial J}\, p(q,J) = dJ\theta(q,J).$$

$$\Rightarrow\; \theta(q,J) = \frac{\partial}{\partial J}\int_0^q dq\, p(q,J) = \frac{\partial}{\partial J}\int_0^q dq\, \sqrt{2m\,[K(J) - V(q)]}.$$

Time evolution: $J = \text{const.}, \quad \theta(t) = \omega(J)t + \theta_0, \quad \omega(J) = \frac{dK}{dJ}.$

$$\Rightarrow\; q(\theta, J) = q(t) \quad \Rightarrow\; p(q,J) = p(t).$$




Generating function of the canonical transformation $(q,p) \to (\theta, J)$:

$$F_2(q,J) = \int_0^q dq\, p(q,J).$$