Dynamical System with 2 Degrees of Freedom

Newton’s equation of motion: \( m\ddot{x} = F(x, \dot{x}), \quad x = (x_1, x_2), \quad y \equiv (\dot{x}_1, \dot{x}_2) \)

Velocity vector field in 4D phase space: \((\dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2)\).

Solution \((x_1(t), x_2(t), y_1(t), y_2(t))\) describes trajectory in 4D phase space. All trajectories are tangential to velocity vector field and nonintersecting.

Orbits are projections of trajectories onto \((x_1, x_2)\)-plane.

Conservative force \( F(x) \): \( \oint d\mathbf{s} \cdot F = 0 \) for all closed paths in \((x_1, x_2)\)-plane.

Potential energy: \( V(x) = -\int^{x}_{x_0} ds \cdot F \).

In conservative system, first integral of the motion guaranteed to exist:
\( E(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + V(x_1, x_2) \).

Existence of second integral of the motion \( K(x_1, x_2, \dot{x}_1, \dot{x}_2) = \text{const} \) guarantees integrability of dynamical system.

In nonintegrable systems, any trajectory in 4D phase space is confined to a 3D hypersurface \( E = \text{const} \). In integrable systems, any trajectory is confined to the intersection of \( E = \text{const} \) and \( K = \text{const} \) in 4D phase space. The resulting 2D manifold has the topology of a torus.

Poincaré surface of section: Plot only those points of a trajectory where it crosses a particular hyperplane (e.g. \( x_1 = 0 \)) in a particular direction (e.g. with \( \dot{x}_1 > 0 \)).

On the Poincaré cut, lines represent quasiperiodic trajectories (on invariant tori) and fixed points represent periodic trajectories.

In integrable systems, the Poincaré maps of all trajectories are confined to lines. In nonintegrable systems, the Poincaré maps of quasiperiodic trajectories are confined to lines whereas the Poincaré maps of chaotic trajectories spread into 2D regions.

Nearby quasiperiodic trajectories move apart linearly in time.
Nearby chaotic trajectories move apart exponentially in time.