Eulerian Angular Velocities

The rotation of a rigid body is described by the vector $\vec{\omega}$ of angular velocity. In general, this vector changes magnitude and direction in both coordinate systems $(x_1, x_2, x_3)$ and $(x'_1, x'_2, x'_3)$.

The most natural formulation of the equations of motion for a rigid body is in the body frame $(x_1, x_2, x_3)$. They are called Euler’s equations.

However, the solution is incomplete unless we know how to express the vector $\vec{\omega}$ in the frame $(x'_1, x'_2, x'_3)$, which is typically the frame of the observer.

Eulerian angular velocities:

- $\dot{\phi}$ directed along $x'_3$-axis.
- $\dot{\theta}$ directed along line of nodes.
- $\dot{\psi}$ directed along $x_3$-axis.

Projections onto axes of $(x_1, x_2, x_3)$:

- $\dot{\psi}_1 = 0, \dot{\psi}_2 = 0, \dot{\psi}_3 = \dot{\psi}$.
- $\dot{\phi}_1 = \dot{\phi} \sin \theta \sin \psi, \dot{\phi}_2 = \dot{\phi} \sin \theta \cos \psi, \dot{\phi}_3 = \dot{\phi} \cos \theta$.

Projections onto axes of $(x'_1, x'_2, x'_3)$:

- $\dot{\phi}'_1 = 0, \dot{\phi}'_2 = 0, \dot{\phi}'_3 = \dot{\phi}$.
- $\dot{\phi}'_1 = \dot{\phi} \sin \theta \sin \phi, \dot{\phi}'_2 = \dot{\phi} \sin \theta \cos \phi, \dot{\phi}'_3 = \dot{\phi} \cos \theta$.

Instantaneous angular velocity in the frame $(x_1, x_2, x_3)$: $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$.

- $\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \psi_1 = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi$.
- $\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \psi_2 = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi$.
- $\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \psi_3 = \dot{\phi} \cos \theta + \dot{\psi}$.

Instantaneous angular velocity in the frame $(x'_1, x'_2, x'_3)$: $\vec{\omega}' = (\omega'_1, \omega'_2, \omega'_3)$.

- $\omega'_1 = \dot{\phi}'_1 + \dot{\theta}'_1 + \psi'_1 = \dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi$.
- $\omega'_2 = \dot{\phi}'_2 + \dot{\theta}'_2 + \psi'_2 = -\dot{\psi} \sin \theta \cos \phi + \dot{\theta} \sin \phi$.
- $\omega'_3 = \dot{\phi}'_3 + \dot{\theta}'_3 + \psi'_3 = \dot{\psi} \cos \theta + \dot{\phi}$.

Magnitude of angular velocity: $|\vec{\omega}|^2 = |\vec{\omega}'|^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi} \dot{\psi} \cos \theta$. 