[mex121] Routhian function of 2D harmonic oscillator

Consider the 2D harmonic oscillator with kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ and potential energy $V = \frac{1}{2}k(x^2 + y^2)$. Expressed in Cartesian coordinates, we have two decoupled systems with one degree of freedom each. The general solution as derived in [mln4] reads,

$$x(t) = x_0 \cos(\omega t), \quad y(t) = y_0 \sin(\omega t + \phi_0), \quad \omega = \sqrt{k/m}. \tag{1}$$

(a) Express the Lagrangian in polar coordinates r, θ, where it is a function $L(r, \dot{r}, \dot{\theta})$, implying that θ is a cyclic coordinate.
(b) Construct the Routhian function $R(r, \dot{r}, \ell)$, where $\ell = \partial L/\partial\dot{\theta}$ is a constant of the motion, readily recognized as the angular momentum.
(c) Derive the equation of motion for the noncyclic coordinate r and an integral expression for the cyclic coordinate θ. Show that $r(t) = \sqrt{x^2(t) + y^2(t)}$ with $x(t)$ and $y(t)$ from (1) is a solution of the former and derive an explicit expression for $\theta(t)$ by evaluating the latter.
(d) Express the (conserved) angular momentum ℓ and energy $E = T+V$ as functions of $m, \omega, x_0, y_0, \phi$.

Solution: