[mex31] Brachistochrone problem II

A particle of mass $m$ slides from rest at the origin of the coordinate system down to the point $E$ along a frictionless path.

(a) Use the calculus of variation to determine the path along which the particle arrives at $E$ in the shortest time $t_E$. Construct an integral expression for $t_E$ with a functional $f[x, x'; y]$. Infer from it the differential expression,

$$dx = \frac{ydy}{\sqrt{2ay - y^2}}, \quad (1)$$

for the optimal path. This expression may not produce a single-valued function $x(y)$.

(b) Circumnavigate the problem with the parametric representation $y(\phi) = a(1 - \cos \phi)$. Infer the dependence $x(\phi)$ from (1) and the dependence $t(\phi)$ from the differential of the original integral expression for $t_E$.

(c) Use the parameters of [mex30], $x_E = 200 \text{m}$, $y_E = 100 \text{m}$, to first determine the values $a$ and $\phi_E$, and then to determine the time $t_E$ it takes the particle to slide along the optimal path. It should be shorter than the time found in [mex30] for the optimal path consisting of two straight segments.

Solution: