

[mex98] Hamilton's principal function for central force problem

Consider the one-body central-force problem specified by the Hamiltonian

$$H(r, p, \ell) = \frac{1}{2m} \left(p^2 + \frac{\ell^2}{r^2} \right) + V(r),$$

where $p \equiv p_r$ and $\ell \equiv p_\vartheta$ are the canonical momenta conjugate to r and ϑ , respectively. Solve the Hamilton-Jacobi equation for Hamilton's principal function. (a) Use the ansatz $S(r, \vartheta, \ell, E, t) = W_1(r, \ell, E) + \ell\vartheta - Et$ for the principal function and determine $W_1(r, \ell, E)$. (b) Infer from $\partial S/\partial E = R = \text{const}$ the time evolution of the radial motion $r(t, E, \ell, r_0)$. (c) Infer from $\partial S/\partial \ell = \Theta = \text{const}$ the orbital relation $\vartheta(r, E, \ell, r_0, \vartheta_0)$, which, in combination with $r(t, E, \ell, r_0)$, determines the time evolution of the angular motion.

Solution: