Partition function and density of states [tln56]

Why do the microcanonical and canonical ensembles yield the same results?

(a) Derivation of Z_N from $\Omega(U,V,N)$.

Relation between the microcanonical phase-space volume $\Omega(U,V,N)$ and the number of microstates $\Sigma(U,V,N)$ up to the energy U:

$$\Omega(U,V,N) \equiv \int_{H(\mathbf{X})<U} d^{6N}X = C_N \Sigma(U,V,N).$$

Density of microstates:

$$g(U) = \frac{\partial \Sigma}{\partial U}.$$

The canonical partition function is then obtained via Laplace transform:

$$\int_0^\infty dU\, g(U) e^{-\beta U} = \frac{1}{C_N} \int_\Gamma d^{6N}X\, e^{-\beta H(\mathbf{X})} = Z_N.$$

Here the energy scale has been shifted such that $U_0 = 0$.

(b) Derivation of $\Omega(U,V,N)$ from Z_N.

Complex continuation of the canonical partition function:

$$Z_N = Z(\beta) \text{ for } \beta = \beta' + i\beta'' \text{ with } \beta' > 0.$$

The microcanonical phase-space volume is the obtained via inverse Laplace transform:

$$g(U) = \frac{1}{2\pi i} \int_{\beta' - i\infty}^{\beta' + i\infty} d\beta\, e^{\beta U} Z(\beta), \quad \Omega(U,V,N) = C_N \int_0^U dU'\, g(U').$$

Both calculations are carried out in exercise [tex81] for the classical ideal gas.