Ideal Bose-Einstein gas: equation of state and internal energy

Conversion of sums into integrals by means of density of energy levels \([\text{tex113}]\):

\[ D(\epsilon) = \frac{V}{\Gamma(D/2)} \left( \frac{m}{2\pi \hbar^2} \right)^{D/2} \epsilon^{D/2-1}, \quad V = L^D. \]

Fundamental thermodynamic relations for BE gas:

\[
\frac{pV}{k_B T} = -\sum_k \ln \left( 1 - ze^{-\beta \epsilon_k} \right) = -\int_0^\infty d\epsilon \ D(\epsilon) \ln \left( 1 - ze^{-\beta \epsilon} \right) = \frac{V}{\lambda_T^D} g_{D/2+1}(z),
\]

\[
N = \sum_k \frac{1}{z-1 e^{\beta \epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)}{z-1 e^{\beta \epsilon} - 1} = \frac{V}{\lambda_T^D} g_{D/2}(z), \quad z < 1,
\]

\[
U = \sum_k \frac{\epsilon_k}{z-1 e^{\beta \epsilon_k} - 1} = \int_0^\infty d\epsilon \frac{D(\epsilon)\epsilon}{z-1 e^{\beta \epsilon} - 1} = \frac{D}{2} k_B T \frac{V}{\lambda_T^D} g_{D/2+1}(z).
\]

The range of fugacity is limited to the interval \(0 \leq z \leq 1\). At \(z = 1\), the expression for \(N\) must, in some cases, be amended by an additive term to account for the possibility of a macroscopic population of the lowest energy level (at \(\epsilon = 0\)).

Equation of state (with fugacity \(z\) in the role of parameter):

\[
\frac{pV}{N k_B T} = \frac{g_{D/2+1}(z)}{g_D/2(z)}, \quad z < 1.
\]