Ideal Fermi-Dirac gas: equation of state and internal energy [tln69]

Conversion of sums into integrals by means of density of energy levels:

$$D(\epsilon) = \frac{gV}{\Gamma(\mathcal{D}/2)} \left(\frac{m}{2\pi\hbar^2}\right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}, \quad V = L^{\mathcal{D}}.$$

Fundamental thermodynamic relations for FD gas:

$$\frac{pV}{k_BT} = \sum_k \ln\left(1 + ze^{-\beta\epsilon_k}\right) = \int_0^\infty d\epsilon\, D(\epsilon) \ln\left(1 + ze^{-\beta\epsilon}\right) = \frac{gV}{\lambda_T^{\mathcal{D}}} f_{\mathcal{D}/2+1}(z),$$

$$\mathcal{N} = \sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} + 1} = \int_0^\infty d\epsilon\, \frac{D(\epsilon)}{z^{-1}e^{\beta\epsilon} + 1} = \frac{gV}{\lambda_T^{\mathcal{D}}} f_{\mathcal{D}/2}(z),$$

$$U = \sum_k \frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k} + 1} = \int_0^\infty d\epsilon\, \frac{D(\epsilon)\epsilon}{z^{-1}e^{\beta\epsilon} + 1} = \frac{\mathcal{D}}{2} k_BT \frac{gV}{\lambda_T^{\mathcal{D}}} f_{\mathcal{D}/2+1}(z).$$

Note: The range of fugacity has no upper limit: $0 \leq z \leq \infty$. The chemical potential μ is unrestricted. The factor g is included to account for any existing level degeneracy due to internal degrees of freedom (e.g. spin) of the fermions.

Equation of state (with fugacity z in the role of parameter):

$$\frac{pV}{\mathcal{N}k_BT} = \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)}.$$