Density of energy levels for ideal quantum gas in $D$ dimensions

(a) Consider a nonrelativistic ideal quantum gas in $D$ dimensions and confined to a box of volume $V = L^D$ with rigid walls. The energy-momentum relation is $\epsilon = \hbar^2 |k|^2 / 2m$ and the density of states in reciprocal space is $\rho(k) = (L/2\pi)^D = \text{const}$. Show that the density of energy levels is

$$D(\epsilon) = \frac{V}{\Gamma(D/2)} \left( \frac{2\pi m \hbar^2}{\epsilon} \right)^{D/2} \epsilon^{D/2-1}.$$

(b) Consider an ultrarelativistic ideal quantum gas in $D$ dimensions and confined to a box of volume $V = L^D$ with rigid walls. The energy-momentum relation is $\epsilon = \hbar |k| c$ and the density of states in reciprocal space is $\rho(k) = (L/2\pi)^D = \text{const}$. Show that the density of energy levels is

$$D(\epsilon) = \frac{V A_D}{(hc)^D} \epsilon^{D-1} = \frac{2V \pi^{D/2}}{\Gamma(D/2)(ch)^D} \epsilon^{D-1}, \quad A_D = \frac{2\pi^{D/2}}{\Gamma(D/2)},$$

where $A_D$ is the area of the unit sphere in $D$ dimensions.

(c) Generalize these results to the case of the relativistic energy-momentum relation,

$$\epsilon = \sqrt{m^2 c^4 + (\hbar kc)^2} - mc^2,$$

and show that the previous two results are recovered in the appropriate limits of the general expression for $D(\epsilon)$.

Solution: