Driven Harmonic Oscillator II _{[\text{mln29}]}

Equation of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A(t)$.

Parameters: $\beta \doteq \gamma/2m$, $\omega_0 \doteq \sqrt{k/m}$, $A(t) \doteq F(t)/m$.

Periodic driving force: $F(t + \tau) = F(t)$.

Fourier series: $A(t) = \dfrac{1}{2}a_0 + \displaystyle\sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$, $\quad \omega = \dfrac{2\pi}{\tau}$.

Fourier coefficients: $a_n = \dfrac{2}{\tau} \displaystyle\int_0^\tau dt\, A(t) \cos n\omega t$, $\quad b_n = \dfrac{2}{\tau} \displaystyle\int_0^\tau dt\, A(t) \sin n\omega t$.

Linear response of system to periodic driving force:

$$x(t) = \frac{a_0}{2\omega_0^2} + \sum_{n=1}^{\infty} d(\omega_n) \left[a_n \cos(\omega_n t + \delta_n) + b_n \sin(\omega_n t + \delta_n)\right],$$

$$\omega_n = \frac{2\pi n}{\tau}, \quad d(\omega_n) = \frac{1}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2\beta^2}}, \quad \delta_n = \arctan\left(\frac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right).$$

Aperiodic driving force: $F(t) = mA(t)$ with $\displaystyle\int_{-\infty}^{+\infty} dt\, |A(t)| < \infty$.

Fourier transform: $\tilde{x}(\omega) = \displaystyle\int_{-\infty}^{+\infty} dt\, e^{i\omega t} x(t)$, $\quad \tilde{A}(\omega) = \displaystyle\int_{-\infty}^{+\infty} dt\, e^{i\omega t} A(t)$.

Inverse transform: $x(t) = \displaystyle\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega)$, $\quad A(t) = \displaystyle\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{A}(\omega)$.

Fourier transformed equation of motion is algebraic (not differential):

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A(t) \;\Rightarrow\; -\omega^2 \tilde{x}(\omega) - 2i\beta\omega\tilde{x}(\omega) + \omega_0^2\tilde{x}(\omega) = \tilde{A}(\omega).$$

Linear response of system to aperiodic driving force:

$$\tilde{x}(\omega) = \frac{\tilde{A}(\omega)}{\omega_0^2 - \omega^2 - 2i\beta\omega} \;\Rightarrow\; x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{x}(\omega).$$