[pex60] Elasticity of balloon during inflation

Consider a rubber balloon in the shape of a sphere during inflation. At some instant during quasistatic inflation the balloon has radius R and the material thickness is h. Upon further inflation the radius increases to $R' = \lambda R$. If we assume that the material is incompressible the thickness must decrease to $h' = \lambda^{-2} h$.
(a) Infer from this information the structure of the deformation gradient tensor $\mathbf{E}(\lambda)$ and the expression for the (tensile) deformation free-energy density $f(\lambda)$ (with guidance from [pln64]).
The total free energy then has two terms, one associated with the elasticity of the rubber material and the other associated with the elasticity of the gas inside the balloon:

$$F_{\text{tot}} = 4\pi R^2 h f(\lambda) - \frac{4\pi}{3} R^3 (\Delta p)(\lambda^3 - 1),$$

where Δp is the excess pressure inside the balloon. The opposite signs of the two terms are consistent with the fact that a change in volume converts one kind of elastic energy into the other kind.
(b) The equilibrium condition that balances the rubber elastic force and pressure is $\partial F_{\text{tot}}/\partial \lambda = 0$. Infer from this condition the dependence of excess pressure Δp on the extent of inflation $\lambda = R'/R$.
(c) Plot the scaled excess pressure $R\Delta p/Gh$ versus λ over the range $1 < \lambda < 3$ and draw your conclusions from the resulting curve.

[adapted from Doi 2013]

Solution: