[pex61] Divergence of stress tensor

Start from the expression,

$$\sigma_{ij} = -p\delta_{ij} + \eta \overbrace{\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_j} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial r_k}\right)}^{\text{zero trace tensor}} + \zeta\,\delta_{ij}\frac{\partial v_k}{\partial r_k},$$

for the stress tensor under the assumption that $\eta = \text{const}$ and $\zeta = \text{const}$ (see [pln86]) and with summation over repeated indices implied. Show that the divergence, $\nabla \cdot \sigma$, of that tensor can then be brought into the form

$$\frac{\partial \sigma_{ij}}{\partial r_j} = -\frac{\partial p}{\partial r_i} + \eta\frac{\partial^2 v_i}{\partial r_j^2} + \beta\eta\frac{\partial}{\partial r_i}\left(\frac{\partial v_j}{\partial r_j}\right), \quad \beta \doteq \frac{\zeta}{\eta} + \frac{1}{3}.$$

[adapted from Bruus 2008]

Solution: