[pex62] Scaled Navier-Stokes equation and the Reynolds number

Start from Navier-Stokes equation for an incompressible fluid in the absence of body forces:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\right] = -\nabla p + \eta\nabla^2\mathbf{v} + \beta\eta\nabla\big(\nabla\cdot\mathbf{v}\big).$$

Introduce scaled variables,

$$\hat{\mathbf{r}} \doteq \frac{\mathbf{r}}{L_0}, \quad \hat{\mathbf{v}} \doteq \frac{\mathbf{v}}{V_0}, \quad \hat{t} \doteq \frac{t}{T_0}, \quad \hat{p} \doteq \frac{p}{P_0},$$

with reference values

$$L_0, \quad V_0, \quad T_0 = \frac{L_0}{V_0}, \quad P_0 = \frac{\eta V_0}{L_0},$$

chosen compatibly. Show that the Navier-Stokes equation expressed in these scaled variables then depends only on a single parameter, which is the Reynolds number Re:

$$Re\left[\frac{\partial}{\partial \hat{t}}\hat{\mathbf{v}} + \big(\hat{\mathbf{v}}\cdot\hat{\nabla}\big)\hat{\mathbf{v}}\right] = -\hat{\nabla}\hat{p} + \hat{\nabla}^2\hat{\mathbf{v}}, \quad Re \doteq \frac{\rho V_0 L_0}{\eta}.$$

[adapted from Bruus 2008]

Solution: