[nex19] Robust probability distributions

Two PDFs and one PMF represent, in turn, a Gaussian, a Lorentz, and a Poisson distribution:

$$\text{(i)}\ \ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-x^2/2\sigma^2}, \qquad \text{(ii)}\ \ f_Y(y) = \frac{1}{\pi}\frac{a}{y^2+a^2}, \qquad \text{(iii)}\ \ P_N(n) = \frac{a^n}{n!}\, e^{-a}.$$

Now consider pairs of independent random variables, (x_1, x_2), (y_1, y_2), and (n_1, n_2), associated with the three distributions, respectively. The distributions of $u = x_1 + x_2$, $v = y_1 + y_2$, and $m = n_1 + n_2$ are then again Gaussian, Lorentzian, and Poissonian in nature, respectively. Most probability distributions do not share this attribute of robustness.
(a) Confirm this assertion by using the following relation and a discrete version thereof:

$$P_U(u) = \int dx_1 \int dx_2\, f_X(x_1) f_X(x_2) \delta(u - x_1 - x_2) = \int dx\, f_X(x) f_X(u - x).$$

(b) Confirm the same assertion by showing that the characteristic functions $\Phi_X(k)$, $\Phi_Y(k)$, and $\Phi_M(k)$, when squared, reproduce the same function, respectively, with modified parameter.

Solution: