[nex34] Random walk in one dimension: unit steps at unit times

Consider the conditional probability distribution $P(n, t_N|0,0)$ describing a biased random walk in one dimension as described by the (discrete) Chapman-Kolmogorov equation,

$$P(n, t_{N+1}|0,0) = \sum_m P(n, t_{N+1}|m, t_N)P(m, t_N|0,0),$$

with discrete times,$t_N = N\tau$, and where

$$P(n, t_{N+1}|m, t_N) = p\delta_{m,n-1} + q\delta_{m,n+1}$$

expresses the instruction that the walker takes a step of unit size forward (with probability p) or backward (with probability $q = 1 - p$) after one time unit τ.
(a) Convert this equation into an equation for the characteristic function,

$$\Phi(k, t_N) = \sum_n e^{ikn}P(n, t_N|0,0),$$

which is readily solved recursively from its initial condition.
(b) Explain how the result,

$$P(n, t_N|0,0) = \frac{N!}{\left(\frac{N+n}{2}\right)!\left(\frac{N-n}{2}\right)!}\, p^{\frac{N+n}{2}} q^{\frac{N-n}{2}},$$

follows most directly from $\Phi(k, t_N)$.

Solution: