Ultracold neutrons in an ideal Steyerl bottle

Consider a container whose walls are perfect mirrors for ultracold neutrons. At time $t = 0$ the bottle is known to contain exactly $n_0$ neutrons. The decay rate of a neutron is $K$.

(a) Set up the master equation,

$$\frac{\partial}{\partial t} P(n, t) = (n + 1)KP(n + 1, t) - KnP(n, t),$$

for the probability distribution $P(n, t)$, and derive the PDE,

$$\frac{\partial G}{\partial t} + K(z - 1)\frac{\partial G}{\partial z} = 0,$$

for the generating function $G(z, t) = \sum_n z^n P(n, t)$.

(b) Solve the PDE by the method of characteristics (see e.g. [nex112]) to obtain

$$G(z, t) = [(z - 1)e^{-Kt} + 1]^{n_0}.$$

(c) Infer therefrom the probability distribution

$$P(n, t) = \frac{n_0!}{n!(n_0 - n)!} \left(\frac{1 - e^{-Kt}}{e^{Kt} - 1}\right)^{n_0}.$$

(d) Determine (via derivatives of the generating function) the average number $\langle n(t) \rangle$ of remaining neutrons and the variance $\langle n^2(t) \rangle$ thereof.

(e) Design a contour plot of $P(n, t)$ for $0 < n < n_0 = 20$ and $0 < Kt < 3$. Design a line graph of $P(n, t)$ for $0 < Kt < 5$ for fixed $n = 0, 1, 2, 5, 10, 15, 18, 19$. Interpret these graphs.

(f) Derive equations of motion for $\langle n \rangle$ and $\langle n(n - 1) \rangle$ directly from the master equation and solve them to reproduce the solutions obtained in (d).

Solution: