[nex8] From Gaussian to exponential distribution

A random variable X has a Gaussian PDF,

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

(a) Confirm the mean $\langle X \rangle = 0$ and the variance $\langle\langle X^2 \rangle\rangle = 1$.
(b) Calculate the PDF $f_Y(y)$ of the random variable Y with values $y = x_1^2 + x_2^2$, where x_1, x_2 are independent realizations of the random variable X, directly from the expression,

$$f_Y(y) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, f_X(x_1) f_X(x_2) \delta(y - x_1^2 - x_2^2).$$

(c) Calculate the same PDF $f_Y(y)$ from the characteristic function,

$$\Phi_Y(k) \doteq \int_{-\infty}^{\infty} dy\, e^{iky} f_Y(y) = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2\, f_X(x_1) f_X(x_2) e^{ik(x_1^2 + x_2^2)},$$

via inverse Fourier transform.
(d) Determine mean $\langle Y \rangle$ and variance $\langle\langle Y^2 \rangle\rangle$.

Solution: