Thermodynamics of Statistically Interacting Quantum Gases

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Fingerprints

Thermodynamic propertiess of quantum gases depend on ...

(1) dimensionality and environment
- energy-momentum relation
- density of 1-particle states

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- perturbation, mean field $\Rightarrow$ modify (1)
- statistical interaction $\Rightarrow$ modify (1) and (2)

Statistical Interaction

Generalized Pauli principle [Haldane 1991]

- Exclusion statistics:

  How does the number of states accessible to a particle depend on the number of particles of the same species already present?

$$\Delta d_i = -g_i \Delta n_i \qquad \Rightarrow \; d_i = A_i - g_i(n_i - 1).$$

  Coefficient of exclusion statistics: $g_i = 0$ (bosons), $g_i = 1$ (fermions).

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- Statistical interaction:

  How is the number of states accessible to one particle of species i affected if other particles (of any species j) are added?

$$\Delta d_i = -\sum_j g_{ij} \Delta n_j \qquad \Rightarrow \; d_i = A_i - \sum_j g_{ij}(n_j - \delta_{ij}),$$

  g_{ij}: statistical interaction coefficients, A_i: statistical capacity constants.

Thermodynamics with Statistical Interaction

Specifications of quantum many-body system:

- bare 1-particle energies: ϵ_i,
- statistical interaction: $A_i,\, g_{ij}$.

Tasks [Wu 1994]:

- combinatorial problem: $W = \prod_i \begin{pmatrix} d_i + n_i - 1 \\ n_i \end{pmatrix}, \quad d_i = A_i - \sum_j g_{ij}(n_j - \delta_{ij})$.
- extremum problem: $\delta(U - TS - \mu\mathcal{N}) = 0$.

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Grand potential: $\Omega = -k_B T \sum_i \ln\left[\frac{1+w_i}{w_i}\right]$ with w_i from

$$\frac{\epsilon_i - \mu}{k_B T} = \ln(1 + w_i) - \sum_j g_{ji} \ln\left(\frac{1+w_i}{w_i}\right).$$

Occupancies $\langle n_i \rangle$ from: $\sum_j (\delta_{ij} w_j + g_{ij}) \langle n_j \rangle = A_i$.

Statistically Interacting Quantum Gas

Nonrelativistic quantum gas in a rigid box of dimensionality $\mathcal{D}$ and volume $V = L^{\mathcal{D}}$.

Specifications:

- Energy-momentum relation: $\epsilon_0(k) = |\mathbf{k}^2|$.
- Statistical interaction: $g(\mathbf{k} - \mathbf{k}')$.

Grand potential: $\Omega(T,V,\mu) = -k_BT\left(\frac{L}{2\pi}\right)^{\mathcal{D}}\int d^{\mathcal{D}}k\,\ln\frac{1+w_{\mathbf{k}}}{w_{\mathbf{k}}}$.

Nonlinear integral equation for $w_{\mathbf{k}}$:

$$\frac{|\mathbf{k}|^2-\mu}{k_BT} = \ln(1+w_{\mathbf{k}}) - \int d^{\mathcal{D}}k'\,g(\mathbf{k}-\mathbf{k}')\ln\frac{1+w_{\mathbf{k}'}}{w_{\mathbf{k}'}}.$$

Linear integral equation for particle density $\langle n_{\mathbf{k}}\rangle$:

$$\langle n_{\mathbf{k}}\rangle w_{\mathbf{k}} + \int d^{\mathcal{D}}k'\,g(\mathbf{k}-\mathbf{k}')\langle n_{\mathbf{k}'}\rangle = 1.$$

Fundamental Thermodynamic Relations

From the solutions $w_{\mathbf{k}}$ and $\langle n_{\mathbf{k}} \rangle$ of the integral equations we calculate. . .

$$\frac{pV}{k_B T} = \left(\frac{L}{2\pi}\right)^{\mathcal{D}} \int d^{\mathcal{D}}k \, \ln \frac{1 + w_{\mathbf{k}}}{w_{\mathbf{k}}}, \tag{1}$$

$$\mathcal{N} = \left(\frac{L}{2\pi}\right)^{\mathcal{D}} \int d^{\mathcal{D}}k \, \langle n_{\mathbf{k}} \rangle, \tag{2}$$

$$U = \left(\frac{L}{2\pi}\right)^{\mathcal{D}} \int d^{\mathcal{D}}k \, |\mathbf{k}|^2 \langle n_{\mathbf{k}} \rangle. \tag{3}$$

(1)+(2): Thermodynamic equation of state

(2)+(3): Caloric equation of state

Dynamically Interacting Quantum Gas

Exactly solvable models in $\mathcal{D} = 1$:

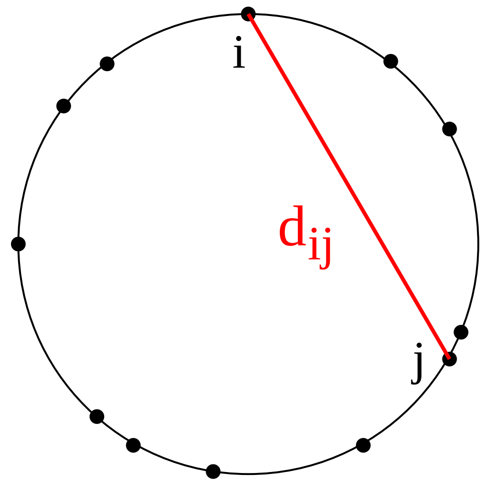
- Nonlinear Schrödinger (NLS) model
  [Lieb and Liniger 1963, Yang and Yang 1969]

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{j<i} \delta(x_i - x_j), \qquad c \geq 0.$$

- Calogero-Sutherland (CS) model
  [Calogero 1971, Sutherland 1971]

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{j<i} \frac{2g(g-1)}{d_{ij}^2}, \qquad 0 \leq g \leq 1,$$

$$d_{ij} = d(x_i - x_j) = \left| \frac{L}{\pi} \sin\left(\frac{\pi}{L}(x_i - x_j) \right) \right|.$$



Equivalence of Dynamical and Statistical Interactions

Thermodynamic Bethe ansatz:

$$\epsilon(k) = k^2 - \mu - \frac{k_B T}{2\pi} \int_{-\infty}^{+\infty} dk' K(k-k') \ln\left(1 + e^{-\epsilon(k')/k_B T}\right) \quad \text{(Yang-Yang eq.)}$$

$$\langle n_k \rangle \left[1 + e^{\epsilon(k)/k_B T}\right] = 1 + \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk' K(k-k') \langle n_{k'} \rangle \quad \text{(Lieb-Liniger eq.)}$$

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Dynamical interaction

- CS model: $K(k-k') = 2\pi(1-g)\delta(k-k')$.
- NLS model: $K(k-k') = \dfrac{2c}{c^2 + (k-k')^2}$.

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Transcription to statistical interaction [Bernard and Wu 1994]:

$$w_k = \exp\left(\frac{\epsilon(k)}{k_B T}\right), \qquad g(k-k') = \delta(k-k') - \frac{1}{2\pi} K(k-k').$$

Generalized CS Model

Statistical interaction: $g(\mathbf{k}-\mathbf{k}') = g\delta(\mathbf{k}-\mathbf{k}')$.

Density of 1-particle states: $D_0(\epsilon_0) = \left(\frac{L}{2\pi}\right)^{\mathcal{D}} \frac{\pi^{\mathcal{D}/2}}{\Gamma(\mathcal{D}/2)}\, \epsilon_0^{\mathcal{D}/2-1}$.

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Grand potential: $\Omega = -k_BT \int_0^\infty d\epsilon_0\, D_0(\epsilon_0) \ln \frac{1+w(\epsilon_0)}{w(\epsilon_0)}$,

$$\frac{1}{z}\exp\left(\frac{\epsilon_0}{k_BT}\right) = [w(\epsilon_0)]^g[1+w(\epsilon_0)]^{1-g}, \quad z = e^{\mu/k_BT}.$$

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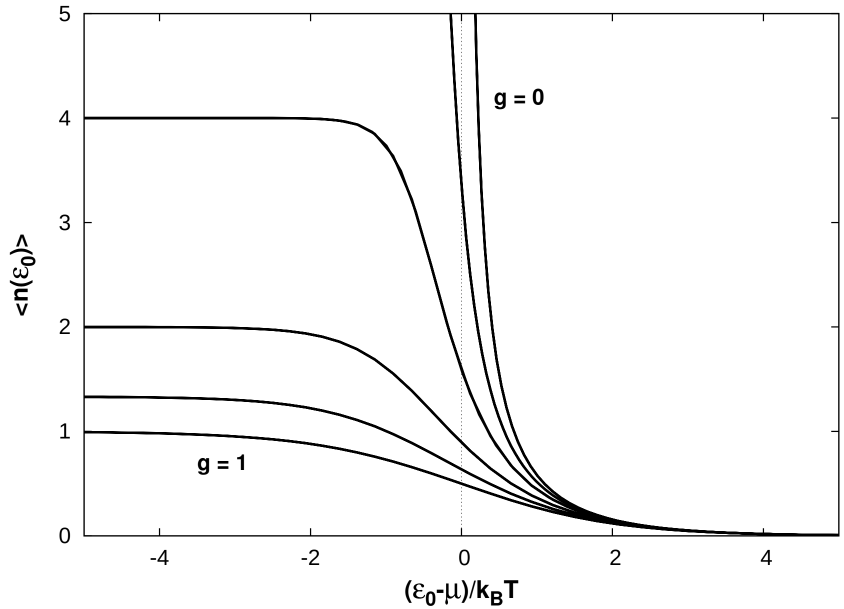
Average level occupancy:

$$\langle n(\epsilon_0)\rangle = \frac{1}{w(\epsilon_0)+g}.$$



CS Functions (1)

Definition for $n > 0$: $G_n(z,g) \doteq \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx\, x^{n-1}}{\bar{w}(x) + g}, \quad [\bar{w}(x)]^g [1 + \bar{w}(x)]^{1-g} = \frac{e^x}{z}.$

Extension to $n \leq 0$: $z \frac{\partial}{\partial z} G_{n+1}(z,g) = G_n(z,g).$

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Fundamental thermodynamic relations:

- $\frac{p\lambda_T^{\mathcal{D}}}{k_B T} = G_{\mathcal{D}/2+1}(z,g),$
- $\frac{\mathcal{N}\lambda_T^{\mathcal{D}}}{V} = G_{\mathcal{D}/2}(z,g) \quad \left[+ \frac{z}{1-z} \quad \text{if} \;\; g=0 \;\; \text{and} \;\; \mathcal{D} > 2 \right],$
- $\frac{U\lambda_T^{\mathcal{D}}}{V} \Big/ \left(\frac{\mathcal{D}}{2} k_B T \right) = G_{\mathcal{D}/2+1}(z,g)$

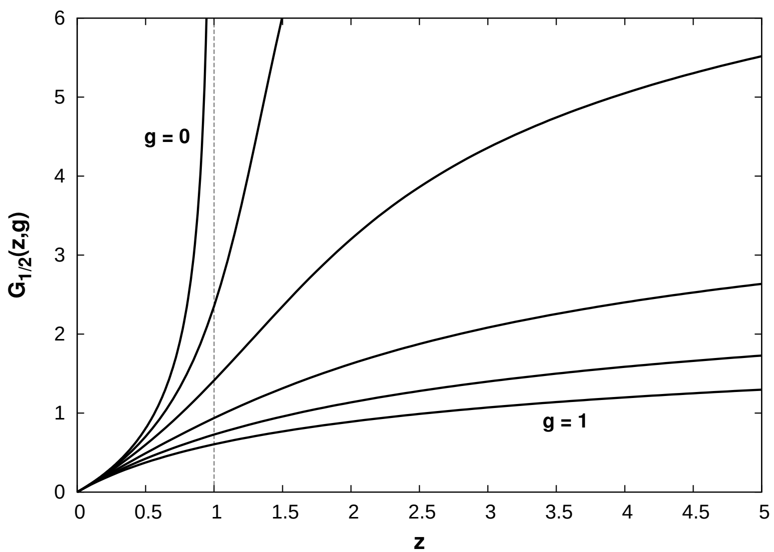
Thermal wavelength: $\lambda_T \doteq \sqrt{\frac{h^2}{2\pi m k_B T}} \stackrel{\hbar^2/2m=1}{\longrightarrow} \sqrt{\frac{4\pi}{k_B T}}.$

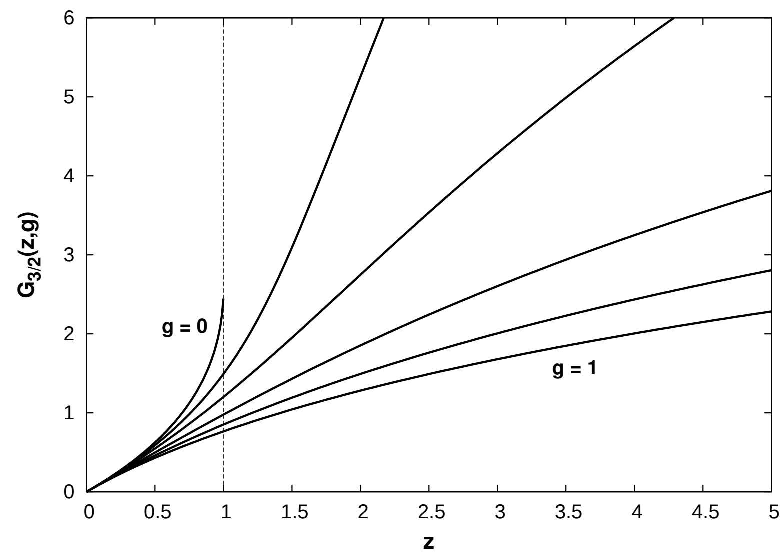
CS Functions (2)

Power series expansion:

$$G_n(z,g) = \sum_{l=1}^{\infty} \frac{z^l}{l^n} \frac{\Gamma(l-lg)}{\Gamma(l)\Gamma(1-lg)}$$
$$= z + \frac{z^2}{2^n}(1-2g) + \frac{z^3}{3^n}(1-3g)\left(1-\frac{3}{2}g\right) + \mathcal{O}(z^4).$$

Asymptotic expansion: $G_n(z,g) \sim \frac{(\ln z)^n}{g\Gamma(n+1)}\left[1 + \frac{\pi^2}{6} gn(n-1)(\ln z)^{-2} + \dots\right].$





Generalized NLS Model

Statistical interaction: $K(\mathbf{k}-\mathbf{k}') = \frac{2\Gamma(\mathcal{D})}{\pi^{\mathcal{D}/2-1}\Gamma(\mathcal{D}/2)} \frac{c^{\mathcal{D}}}{[c^2+(\mathbf{k}-\mathbf{k}')^2]^{\mathcal{D}}}.$

$$\lim_{c\to\infty} K(\mathbf{k}) = 0, \quad \lim_{c\to 0} K(\mathbf{k}) = 2\pi\delta(\mathbf{k}), \quad \int d^{\mathcal{D}}k\, K(\mathbf{k}) = 2\pi.$$

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Generalized Yang-Yang equation:

$$\bar{\epsilon}(\bar{k}) = \bar{k}^2 - \ln z - \frac{2\bar{c}^{\mathcal{D}}\Gamma(\mathcal{D})}{[\Gamma(\mathcal{D}/2)]^2} \int_0^\infty \frac{d\bar{k}'\,\bar{k}'^{\mathcal{D}-1}(\bar{c}^2+\bar{k}^2+\bar{k}'^2)\ln(1+e^{-\bar{\epsilon}(\bar{k}')})}{[\bar{c}^4+2\bar{c}^2(\bar{k}^2+\bar{k}'^2)+(\bar{k}^2-\bar{k}'^2)^2]^{(\mathcal{D}+1)/2}}.$$

$$\bar{k} \doteq \frac{k}{\sqrt{k_BT}}, \quad \bar{\epsilon}(\bar{k}) \doteq \frac{\epsilon(k)}{k_BT}, \quad \bar{n}(\bar{k}) \doteq n(k), \quad \bar{c} \doteq \frac{c}{\sqrt{k_BT}}.$$

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Generalized Lieb-Liniger equation:

$$\bar{n}(\bar{k})\left[1+e^{\bar{\epsilon}(\bar{k})}\right] = 1 + \frac{2\bar{c}^{\mathcal{D}}\Gamma(\mathcal{D})}{[\Gamma(\mathcal{D}/2)]^2} \int_0^\infty \frac{d\bar{k}'\,\bar{k}'^{\mathcal{D}-1}(\bar{c}^2+\bar{k}^2+\bar{k}'^2)\bar{n}(\bar{k}')}{[\bar{c}^4+2\bar{c}^2(\bar{k}^2+\bar{k}'^2)+(\bar{k}^2-\bar{k}'^2)^2]^{(\mathcal{D}+1)/2}}.$$

NLS Functions

Fundamental thermodynamic relations:

- $$\frac{p\lambda_T^{\mathcal{D}}}{k_B T} = \frac{2}{\Gamma(\mathcal{D}/2)} \int_0^\infty d\bar{k}\, \bar{k}^{\mathcal{D}-1} \ln\left(1 + e^{-\bar{\epsilon}(\bar{k})}\right) \doteq F_p^{(\mathcal{D})}(z, \bar{c}),$$
- $$\frac{\mathcal{N}\lambda_T^{\mathcal{D}}}{V} = \frac{2}{\Gamma(\mathcal{D}/2)} \int_0^\infty d\bar{k}\, \bar{k}^{\mathcal{D}-1} \bar{n}(\bar{k}) \doteq F_{\mathcal{N}}^{(\mathcal{D})}(z, \bar{c}),$$
- $$\frac{U\lambda_T^{\mathcal{D}}}{V} \Big/ \frac{\mathcal{D}}{2} k_B T = \frac{2}{\Gamma(\mathcal{D}/2+1)} \int_0^\infty d\bar{k}\, \bar{k}^{\mathcal{D}+1} \bar{n}(\bar{k}) \doteq F_U^{(\mathcal{D})}(z, \bar{c}).$$

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Note:

- NLS functions depend separately on μ and T
- BE functions recovered for $\bar{c}=0$
- FD functions recovered for $\bar{c}=\infty$.
- $F_p^{(\mathcal{D})}(z,\bar{c}) \neq F_U^{(\mathcal{D})}(z,\bar{c})$.

Thermodynamic Equation of State

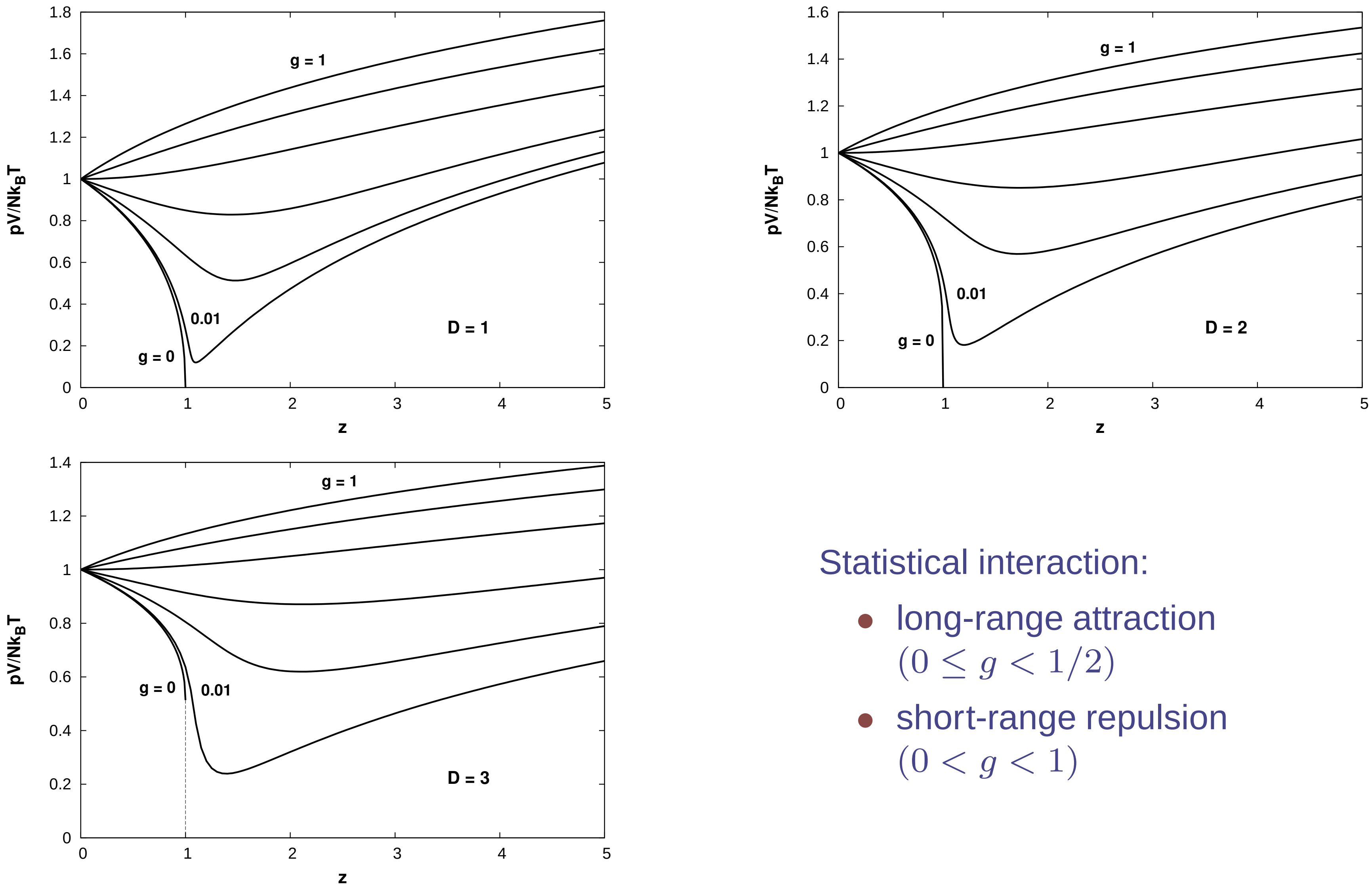
Universal functional relations between p, T, and $v \doteq V/\mathcal{N}$:

- Isochores: $\dfrac{p}{p_v} = \dfrac{G_{\mathcal{D}/2+1}(z,g)}{[G_{\mathcal{D}/2}(z,g)]^{1+2/\mathcal{D}}}, \quad \dfrac{T}{T_v} = [G_{\mathcal{D}/2}(z,g)]^{-2/\mathcal{D}}.$
- Isotherms: $\dfrac{p}{p_T} = G_{\mathcal{D}/2+1}(z,g), \quad \dfrac{v}{v_T} = \left[G_{\mathcal{D}/2}(z,g)\right]^{-1}.$
- Isobars: $\dfrac{v}{v_p} = \dfrac{[G_{\mathcal{D}/2+1}(z,g)]^{\mathcal{D}/(\mathcal{D}+2)}}{G_{\mathcal{D}/2}(z,g)}, \quad \dfrac{T}{T_p} = \left[G_{\mathcal{D}/2+1}(z,g)\right]^{-2/(\mathcal{D}+2)}.$

Reference values from $pv = k_BT$ and $v = \lambda_T^{\mathcal{D}}$.

Equation of state: $\dfrac{pV}{\mathcal{N}k_BT} = \dfrac{G_{\mathcal{D}/2+1}(z,g)}{G_{\mathcal{D}/2}(z,g)}.$

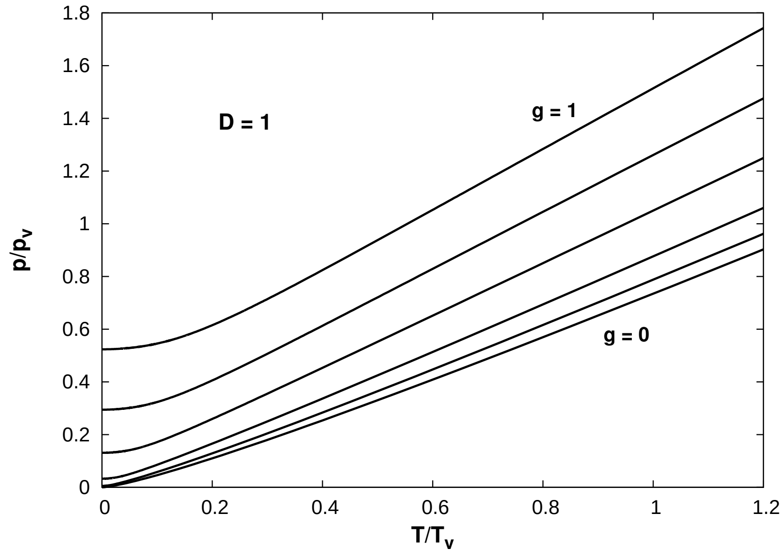
Equation of State

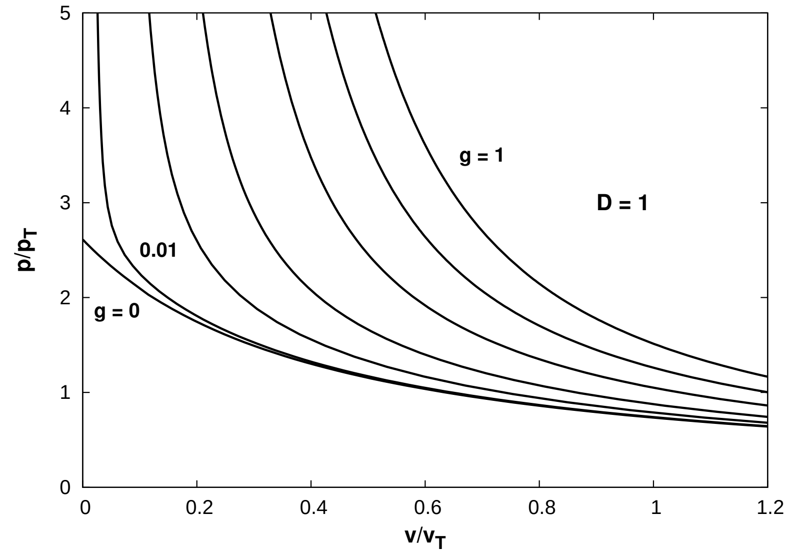


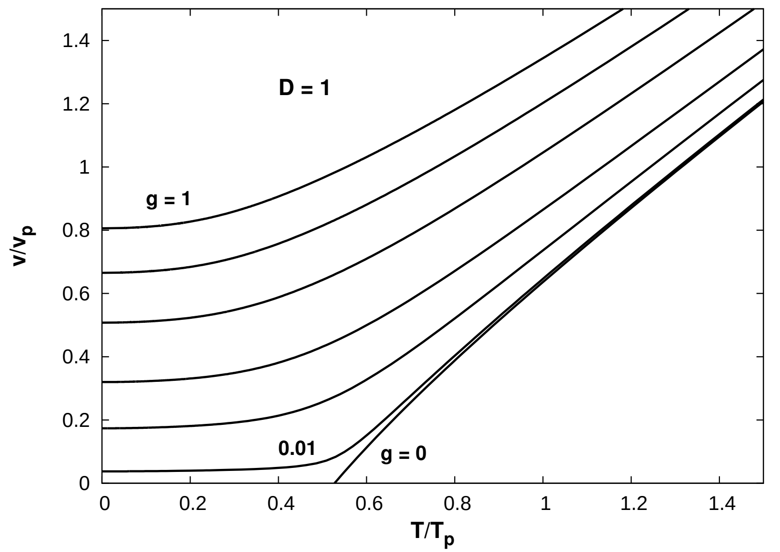
Statistical interaction:

- long-range attraction
  $(0 \leq g < 1/2)$
- short-range repulsion
  $(0 < g < 1)$

CS Model in $\mathcal{D} = 1$







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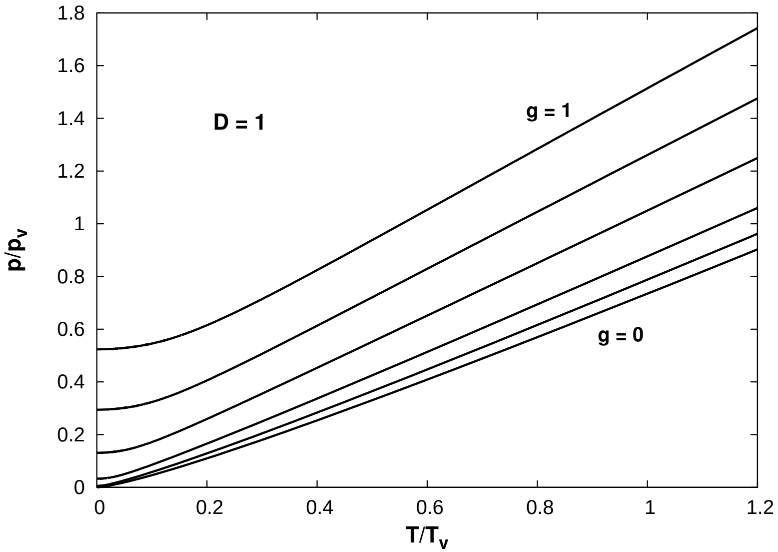
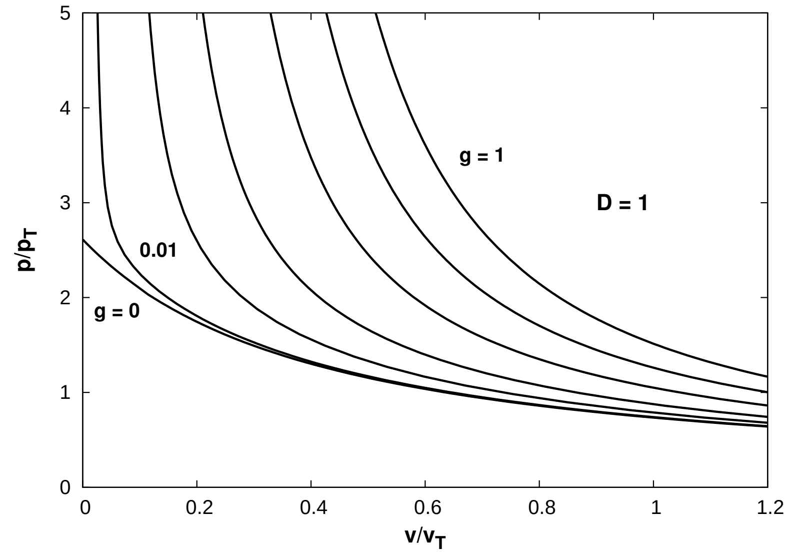
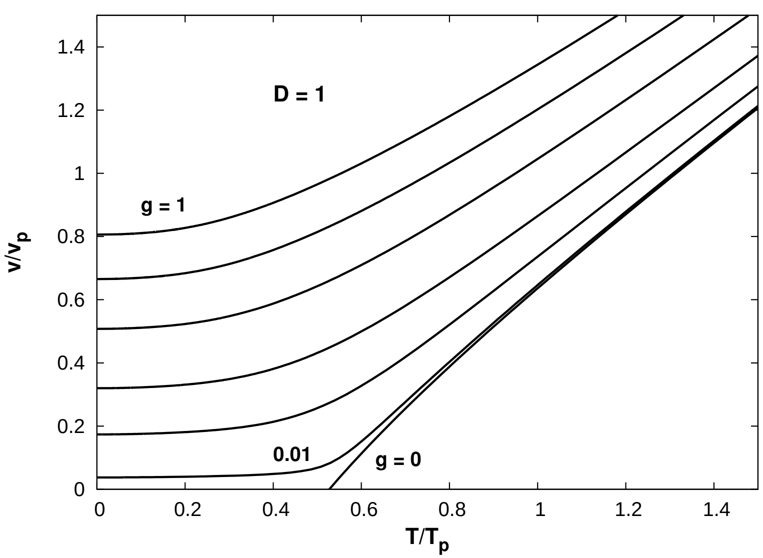
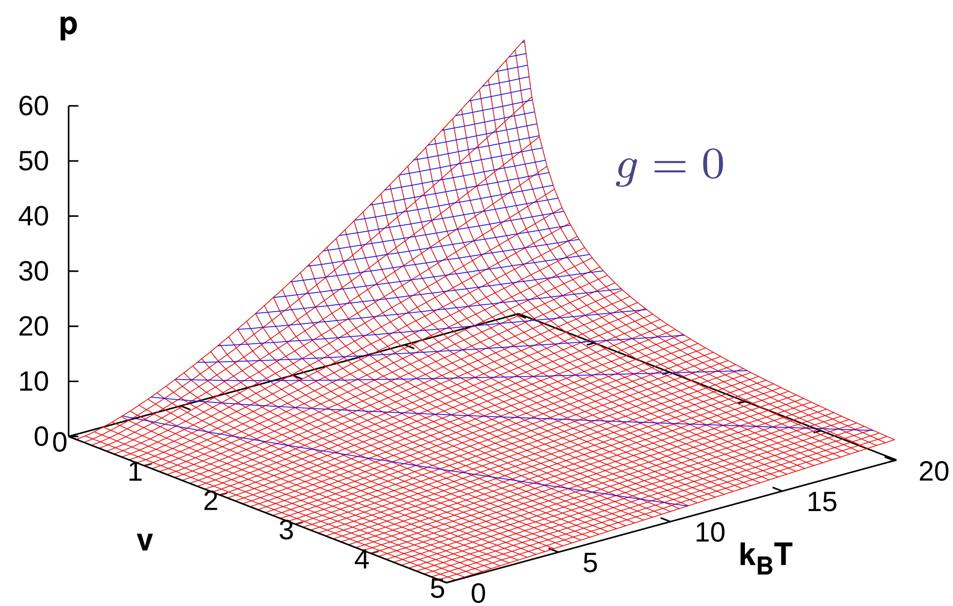
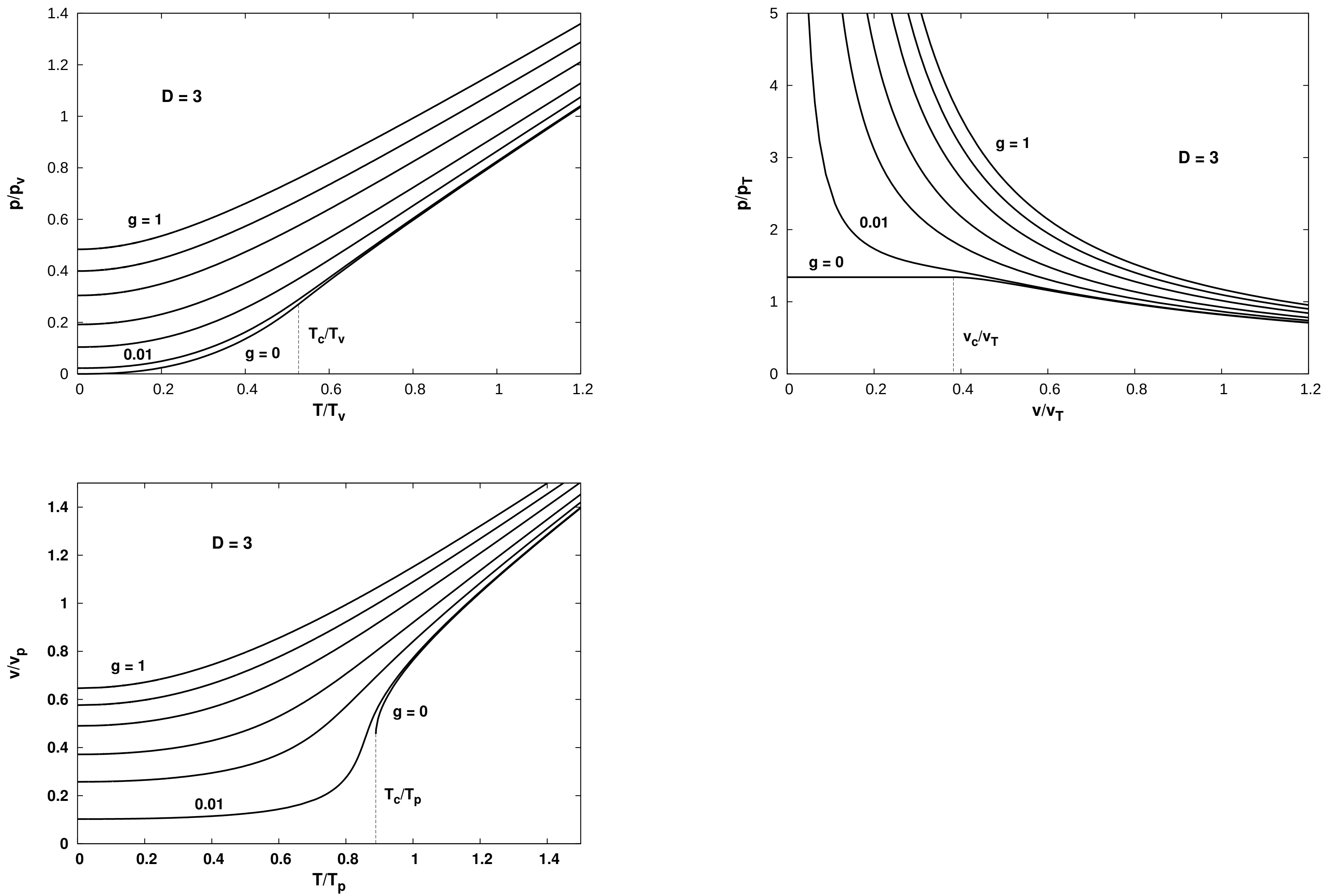








CS Model in $\mathcal{D} = 3$



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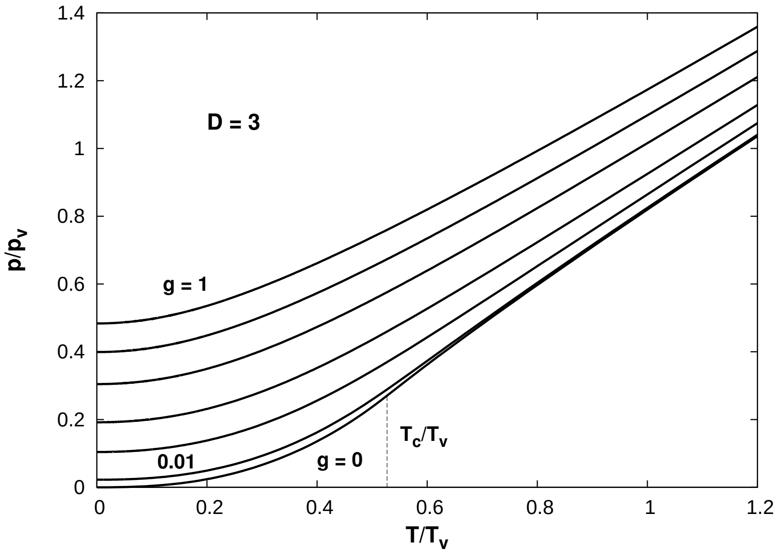
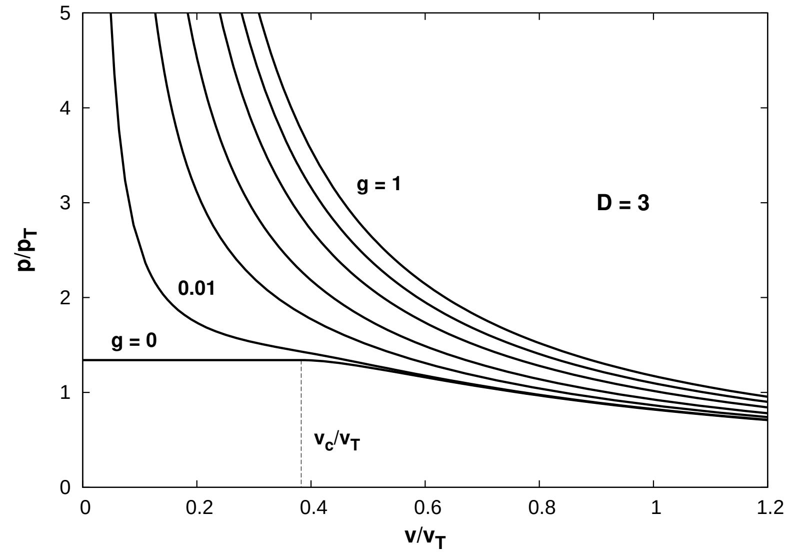
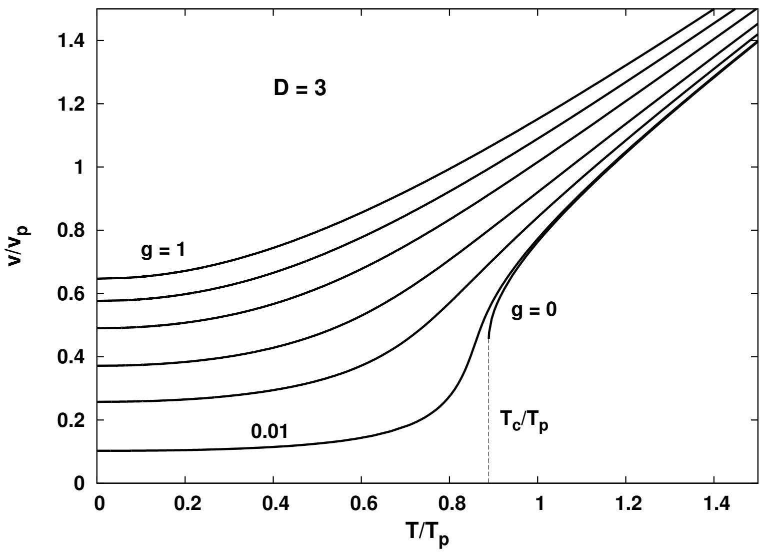
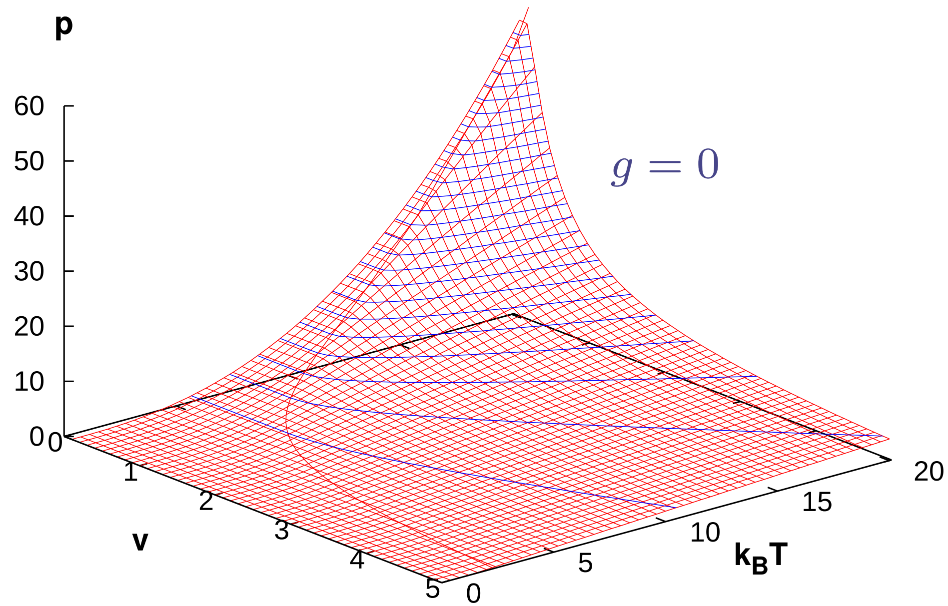






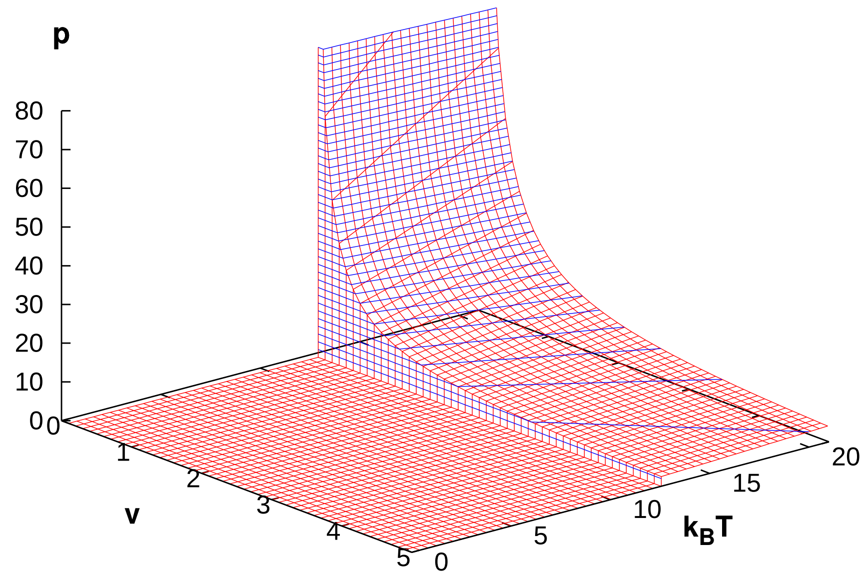


BE and FD Gases in $\mathcal{D} = \infty$

Ideal BE gas in $\mathcal{D} = \infty$:

$$pv = \begin{cases} k_B T, & T > T_c \quad \text{(ideal MB gas)} \\ 0, & T < T_c \quad \text{(pure condensate)} \end{cases}$$

$$k_B T_c = 4\pi.$$



BE and FD Gases in $\mathcal{D} = \infty$

Ideal BE gas in $\mathcal{D} = \infty$:

$$pv = \begin{cases} k_B T, & T > T_c \quad \text{(ideal MB gas)} \\ 0, & T < T_c \quad \text{(pure condensate)} \end{cases}$$

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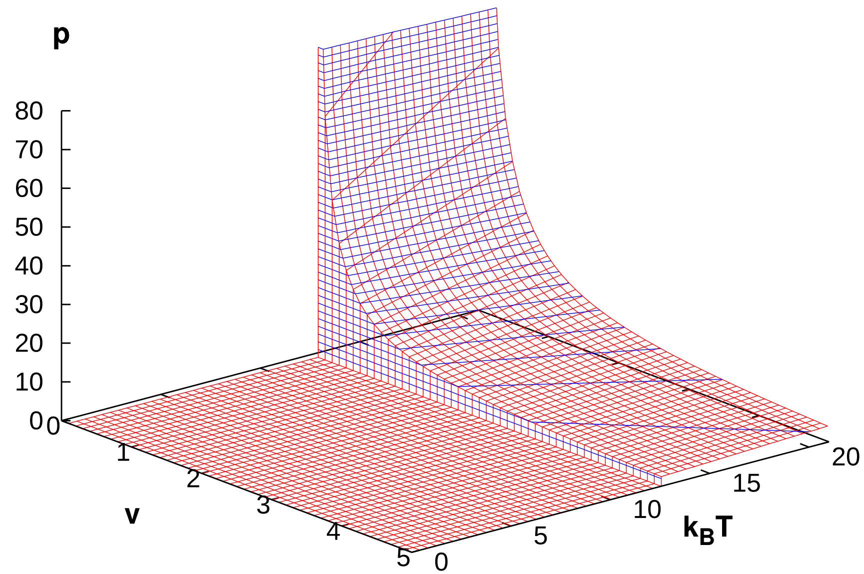
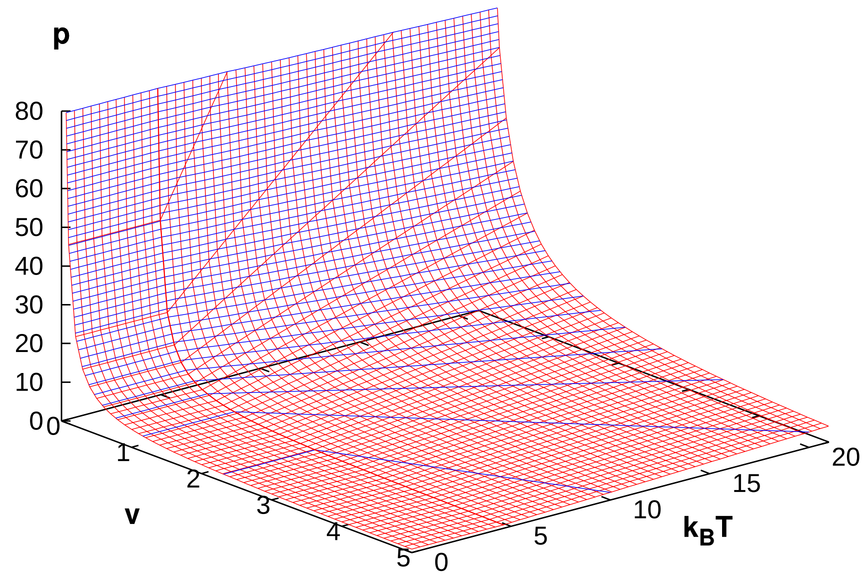


Ideal FD gas in $\mathcal{D} = \infty$:

$$pv = \begin{cases} k_B T, & T > T_c \quad \text{(ideal MB gas)} \\ k_B T_c, & T < T_c \quad \text{(pure Fermi sea)} \end{cases}$$

$$k_B T_c = \frac{4\pi}{e}.$$



Free fermions in $\mathcal{D} = \infty$

Non-commuting limits $z \to \infty, \ \mathcal{D} \to \infty$:

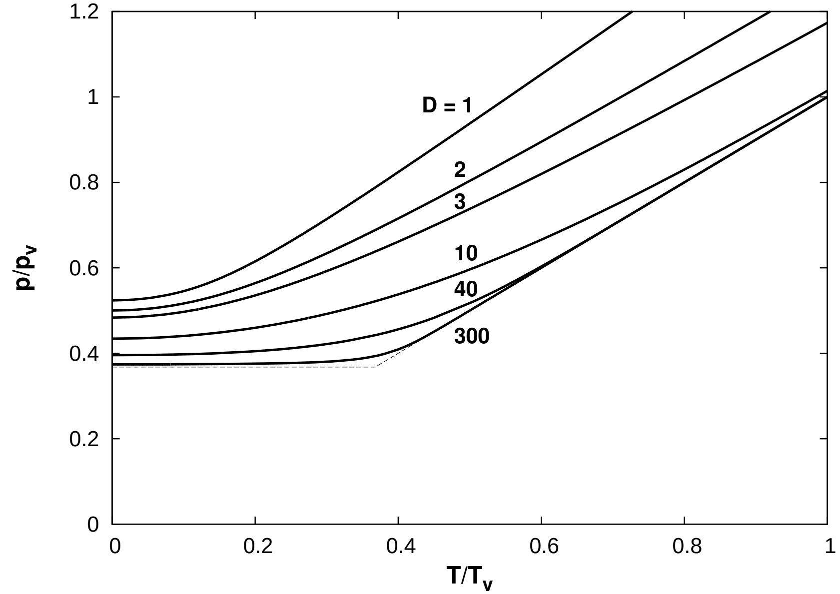
- $z < \infty, \ \mathcal{D} \to \infty$:

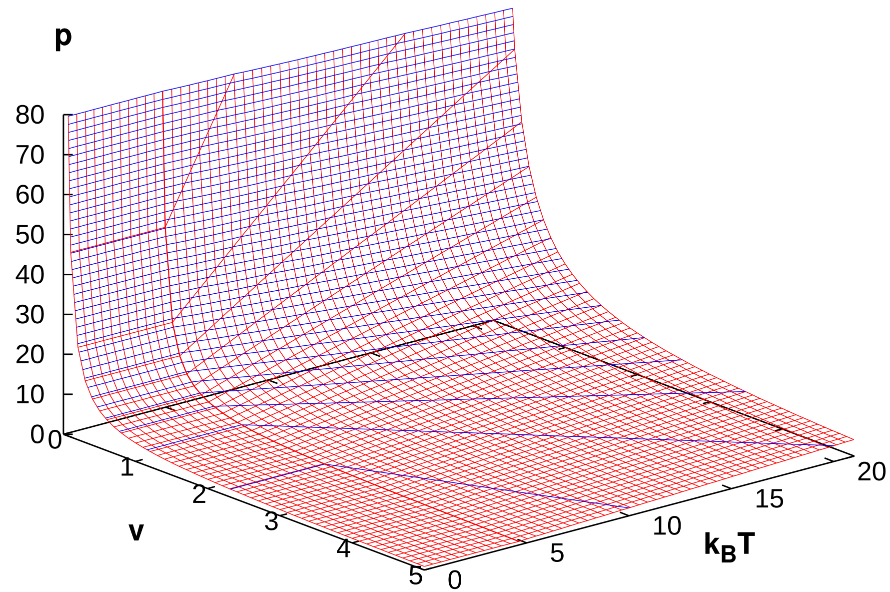
$$\frac{p}{p_v} = \frac{T}{T_v} \frac{f_{\mathcal{D}/2+1}(z)}{f_{\mathcal{D}/2}(z)} \xrightarrow{\mathcal{D}\to\infty} \frac{T}{T_v}.$$

- $\mathcal{D} \to \infty, \ z \to \infty$ with $\mathcal{D}/2 = r \ln z, \ r \geq 0$:

$$\frac{p}{p_v} = \frac{f_{\mathcal{D}/2+1}(z)}{[f_{\mathcal{D}/2}(z)]^{1+2/\mathcal{D}}} \overset{\mathcal{D}\gg 1}{\rightsquigarrow} \frac{e^{-1}}{1+2/\mathcal{D}},$$

$$\frac{T}{T_v} = [f_{\mathcal{D}/2}(z)]^{-2/\mathcal{D}} \overset{\mathcal{D}\gg 1}{\rightsquigarrow} \frac{\mathcal{D}}{2} \frac{e^{-1}}{\ln z},$$





Response functions

- Isochoric heat capacity $C_v \doteq \mathcal{N}^{-1}\left(\frac{\partial U}{\partial T}\right)_V$:

$$\frac{C_v}{k_B} = \frac{\mathcal{D}}{2}\left[\left(\frac{\mathcal{D}}{2}+1\right)\frac{G_{\mathcal{D}/2+1}(z,g)}{G_{\mathcal{D}/2}(z,g)} - \frac{\mathcal{D}}{2}\frac{G_{\mathcal{D}/2}(z,g)}{G_{\mathcal{D}/2-1}(z,g)}\right].$$

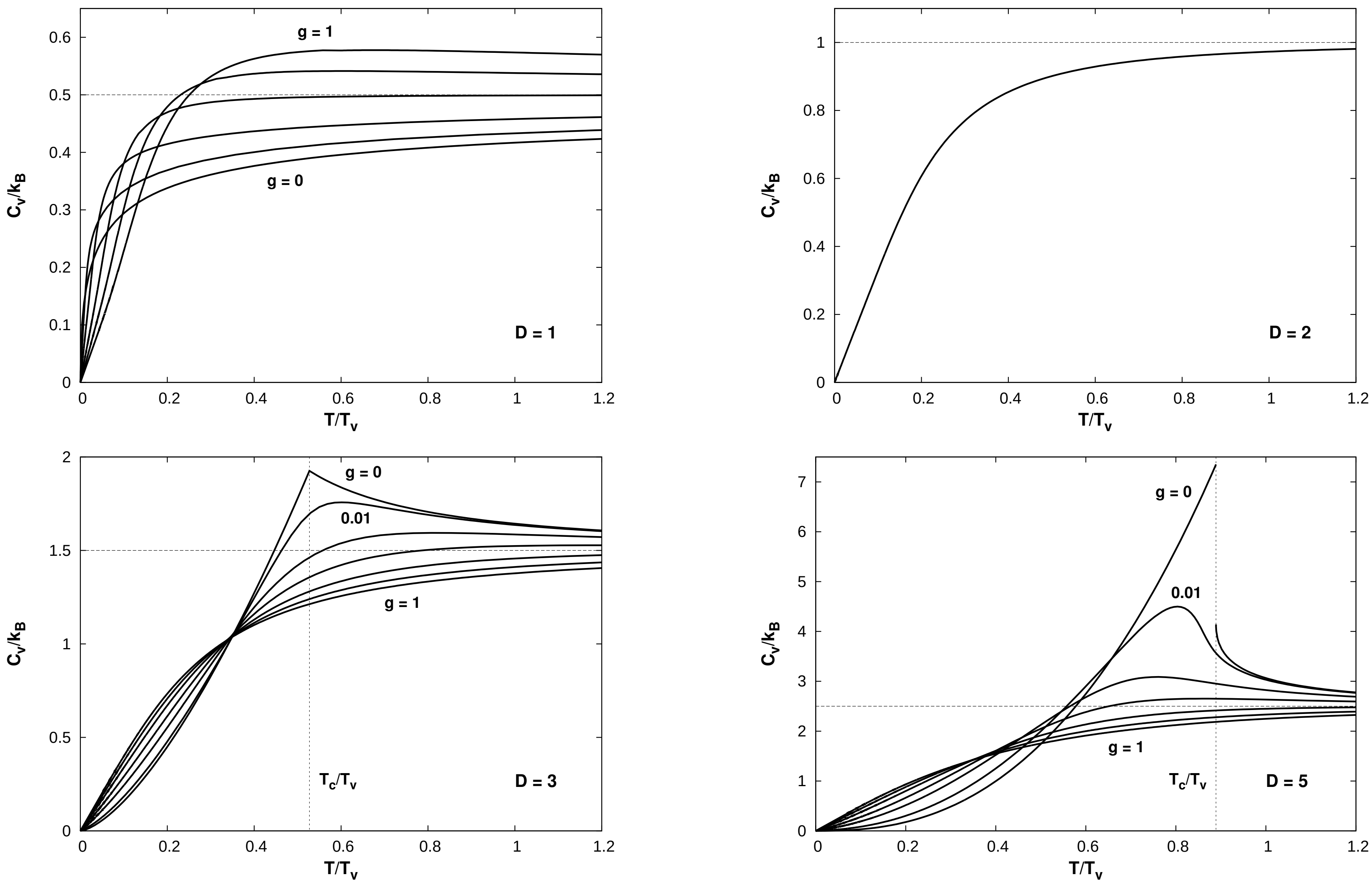
- Isothermal compressibility $\kappa_T \doteq -v^{-1}\left(\frac{\partial v}{\partial p}\right)_T$:

$$p_T\kappa_T = \frac{G_{\mathcal{D}/2-1}(z,g)}{[G_{\mathcal{D}/2}(z,g)]^2}.$$

- Isobaric expansivity $\alpha_p \doteq v^{-1}\left(\frac{\partial v}{\partial T}\right)_p$:

$$T_p\alpha_p = \frac{T_p}{T}\left[\left(\frac{\mathcal{D}}{2}+1\right)\frac{G_{\mathcal{D}/2+1}(z,g)G_{\mathcal{D}/2-1}(z,g)}{G_{\mathcal{D}/2}(z,g)G_{\mathcal{D}/2}(z,g)} - \frac{\mathcal{D}}{2}\right].$$

Isochoric Heat Capacity



Velocity of Sound

Result from transport theory: $c = \dfrac{1}{\sqrt{\rho\kappa_S}}$.

Relation to quantities previously determined:

$$\frac{mc^2}{k_B T} = \frac{(v/v_T)}{(p_T\kappa_T)}\left[1 + \frac{(T/T_p)^2(v/v_T)(T_p\alpha_p)^2}{(p_T\kappa_T)(C_v/k_B)}\right].$$

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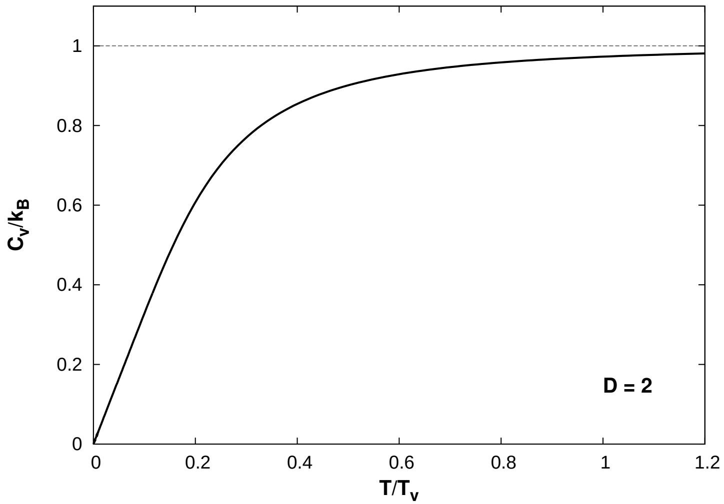
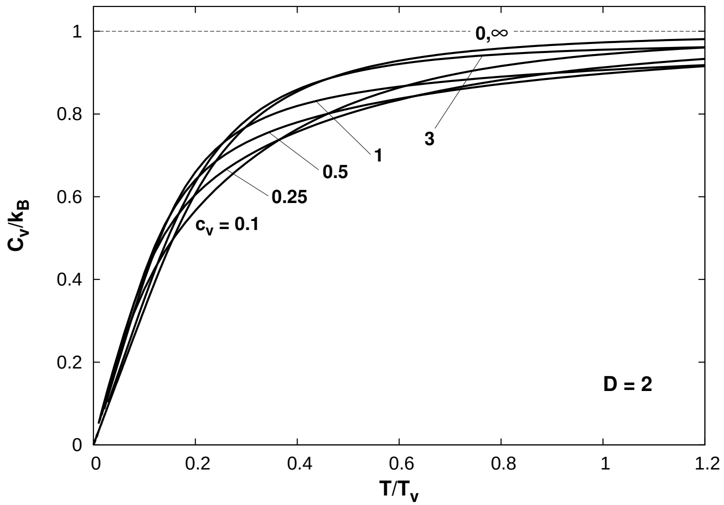
CS model: $\dfrac{mc^2}{k_BT} = \gamma\,\dfrac{G_{\mathcal{D}/2+1}(z,g)}{G_{\mathcal{D}/2}(z,g)}, \qquad \gamma \doteq 1 + \dfrac{2}{\mathcal{D}}.$

- $\dfrac{mc^2}{k_BT_v} = \gamma\,\dfrac{p}{p_v} \qquad (v = \text{const.}),$

- $\dfrac{mc^2}{k_BT_p} = \gamma\,\dfrac{v}{v_p} \qquad (p = \text{const.}),$

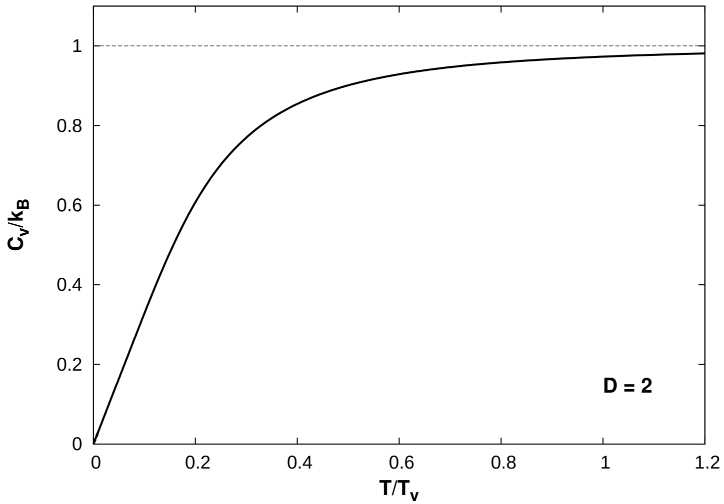
Heat capacity in $\mathcal{D} = 2$

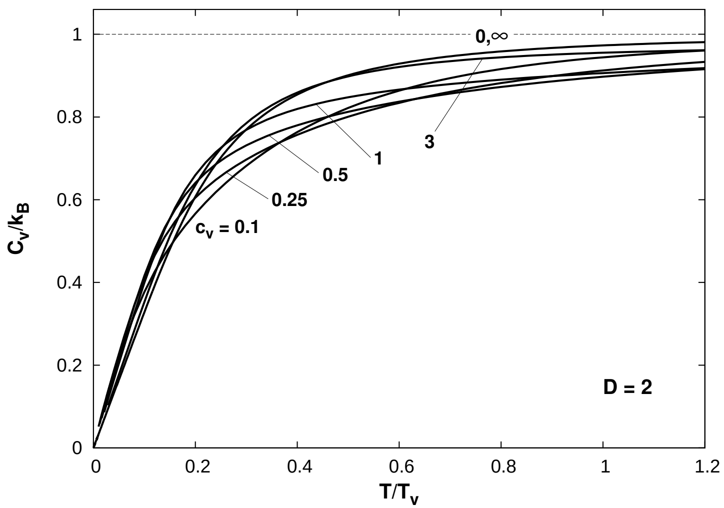




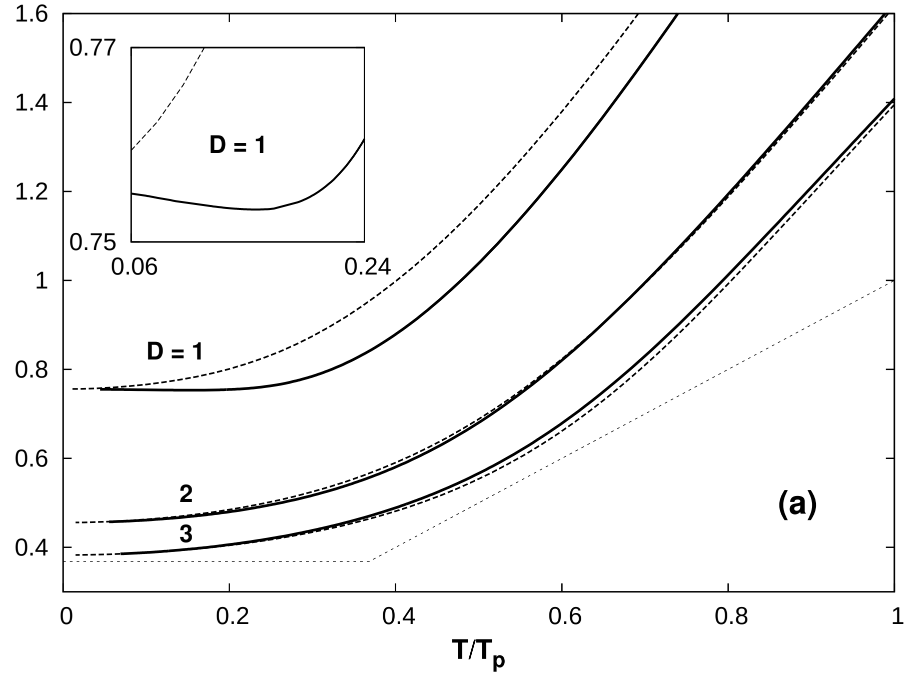
Statistical Interaction Versus Fractional Statistics

Heat capacity in $\mathcal{D} = 2$





Speed of sound for $p = \text{const.}$



mc^2/k_BT versus T/T_p (solid lines)

$\gamma v/v_p$ versus T/T_p (dashed lines)

Shell Model

Statistical interaction: $g(\mathbf{k}-\mathbf{k}') = \dfrac{g\Gamma(\mathcal{D}/2)}{2\pi^{\mathcal{D}/2}Q^{\mathcal{D}-1}}\delta(|\mathbf{k}-\mathbf{k}'|-Q).$

$$\int d^{\mathcal{D}}k'\, g(\mathbf{k}-\mathbf{k}') = g, \qquad \lim_{Q\to 0} g(\mathbf{k}-\mathbf{k}') = g\delta(\mathbf{k}-\mathbf{k}').$$

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$\mathcal{D}=1 \quad \Rightarrow$ coupled algebraic equations:

$$\frac{k^2-\mu}{k_BT} = \ln(1+w_k) - \frac{1}{2}g\left[\ln\frac{1+w_{k-Q}}{w_{k-Q}} + \ln\frac{1+w_{k+Q}}{w_{k+Q}}\right],$$

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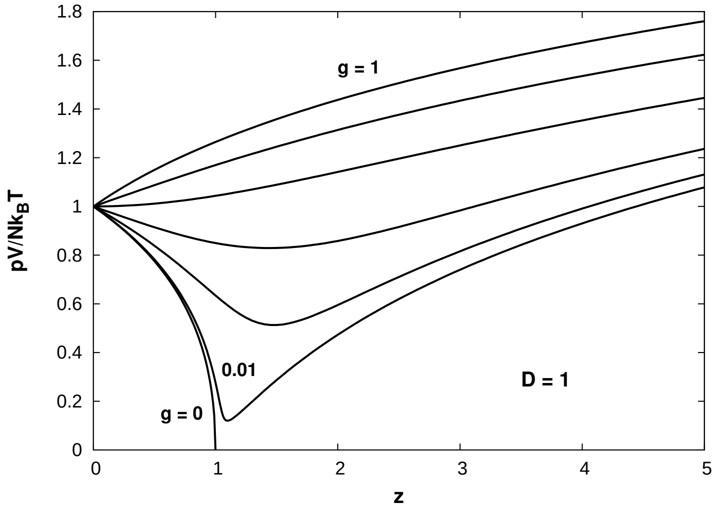
$\mathcal{D}=2 \quad \Rightarrow$ integral equations:

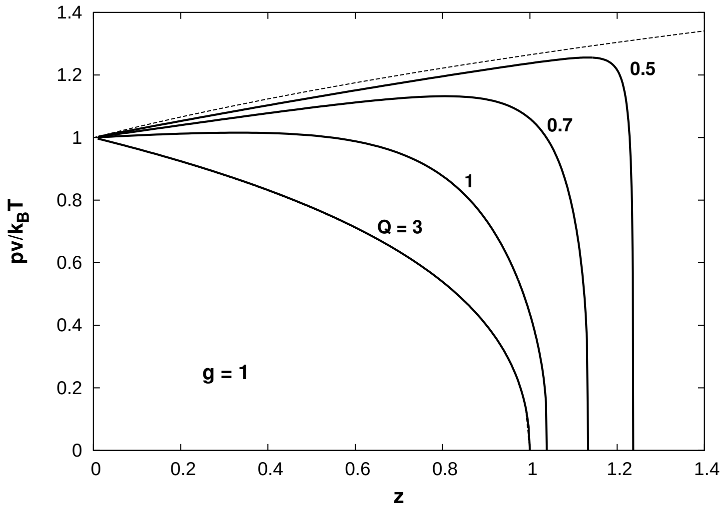
$$\frac{k^2-\mu}{k_BT} = \ln(1+w_k) - \frac{g}{2\pi}\int_0^{2\pi} d\phi\, \ln\frac{1+w_{k'}}{w_{k'}},$$

$$\langle n_k\rangle = \frac{1}{w_k}\left[1 - \frac{g}{2\pi}\int_0^{2\pi} d\phi\,\langle n_{k'}\rangle\right], \qquad k' = \sqrt{k^2-2kQ\cos\phi+Q^2}.$$

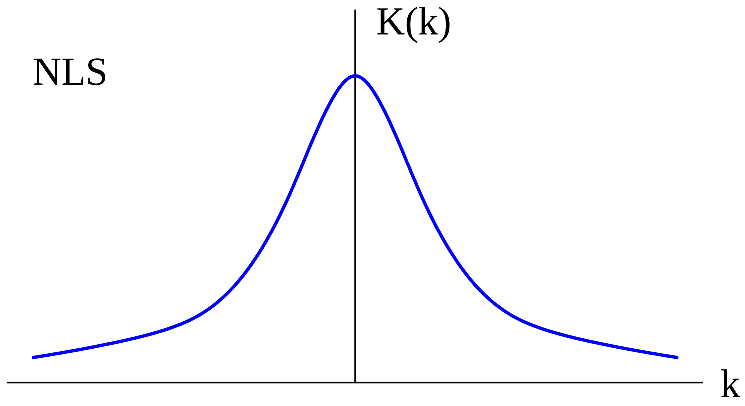
Bosonic and Fermionic Features

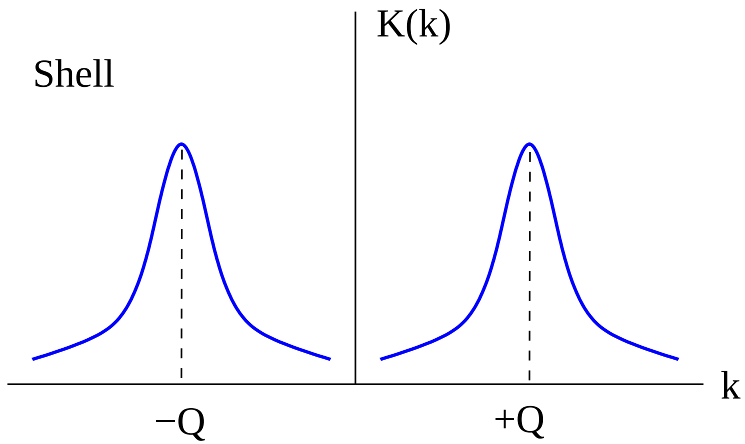
Equation of state in $\mathcal{D} = 1$





Statistical interaction kernel $K(k)$





Publications

- G. G. Potter, G. Müller, and M. Karbach
  Thermodynamics of ideal quantum gas with fractional statistics in D dimensions
  Phys. Rev. E 75, 061120 (2007)
  [arXiv:cond-mat/0610400]

- G. G. Potter, G. Müller, and M. Karbach
  Thermodynamics of statistically interacting quantum gas in D dimensions
  Phys. Rev. E 76, 061112 (2007)
  [arXiv:0710.1031]

Virial Coefficients

Virial expansion: $$\frac{pv}{k_BT} = 1 + \sum_{\ell=2}^{\infty} a_\ell \left(\frac{\lambda_T^{\mathcal{D}}}{v}\right)^{\ell-1}; \qquad \lambda_T \doteq \sqrt{\frac{h^2}{2\pi m k_B T}}, \quad v \doteq \frac{V}{\mathcal{N}}.$$

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- Shell model: $a_2 = \dfrac{1}{2^{\mathcal{D}/2+1}}\left[1 - 2g\exp\left(-\dfrac{Q_T^2}{2}\right)\right]$, $Q_T \doteq \dfrac{Q}{\sqrt{k_BT}}$.

Reference Values

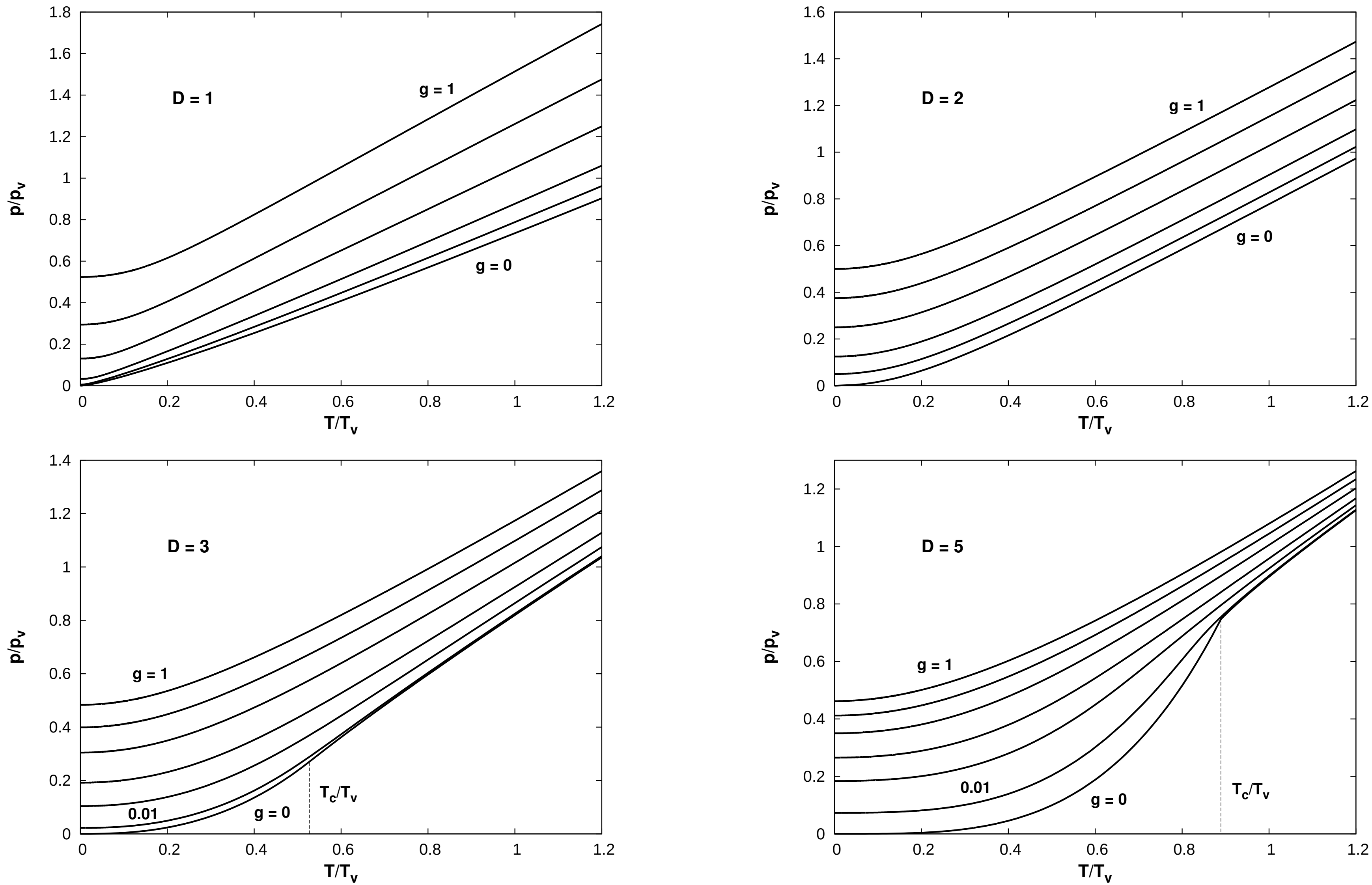
Basis: $v = \lambda_T^{\mathcal{D}}$ and MB equation of state.

- $k_BT_v = \dfrac{4\pi}{v^{2/\mathcal{D}}}, \quad p_v = \dfrac{4\pi}{v^{2/\mathcal{D}+1}} \qquad (v = \text{const.})$
- $v_T = \left(\dfrac{4\pi}{k_BT}\right)^{\mathcal{D}/2}, \quad p_T = 4\pi\left(\dfrac{k_BT}{4\pi}\right)^{\mathcal{D}/2+1} \qquad (T = \text{const.})$
- $k_BT_p = 4\pi\left(\dfrac{p}{4\pi}\right)^{\frac{2}{\mathcal{D}+2}}, \quad v_p = \left(\dfrac{4\pi}{p}\right)^{\frac{\mathcal{D}}{\mathcal{D}+2}} \qquad (p = \text{const.})$

Basis: $\ln z \doteq \bar{T}_v/T$ or $\ln z \doteq \bar{T}_p/T$ and FD equation of state.

- $\dfrac{\bar{T}_v}{T_v} = \dfrac{\bar{p}_v}{p_v} = \left[\Gamma\left(\dfrac{\mathcal{D}}{2}+1\right)\right]^{\frac{2}{\mathcal{D}}} \overset{\mathcal{D}\gg 1}{\rightsquigarrow} \dfrac{\mathcal{D}}{2e} \qquad (v = \text{const.})$
- $\dfrac{\bar{T}_p}{T_p} = \dfrac{\bar{v}_p}{v_p} = \left[\Gamma\left(\dfrac{\mathcal{D}}{2}+2\right)\right]^{\frac{2}{\mathcal{D}+2}} \overset{\mathcal{D}\gg 1}{\rightsquigarrow} \left(\dfrac{\mathcal{D}}{2}+1\right)\dfrac{1}{e} \qquad (p = \text{const.})$

Isochores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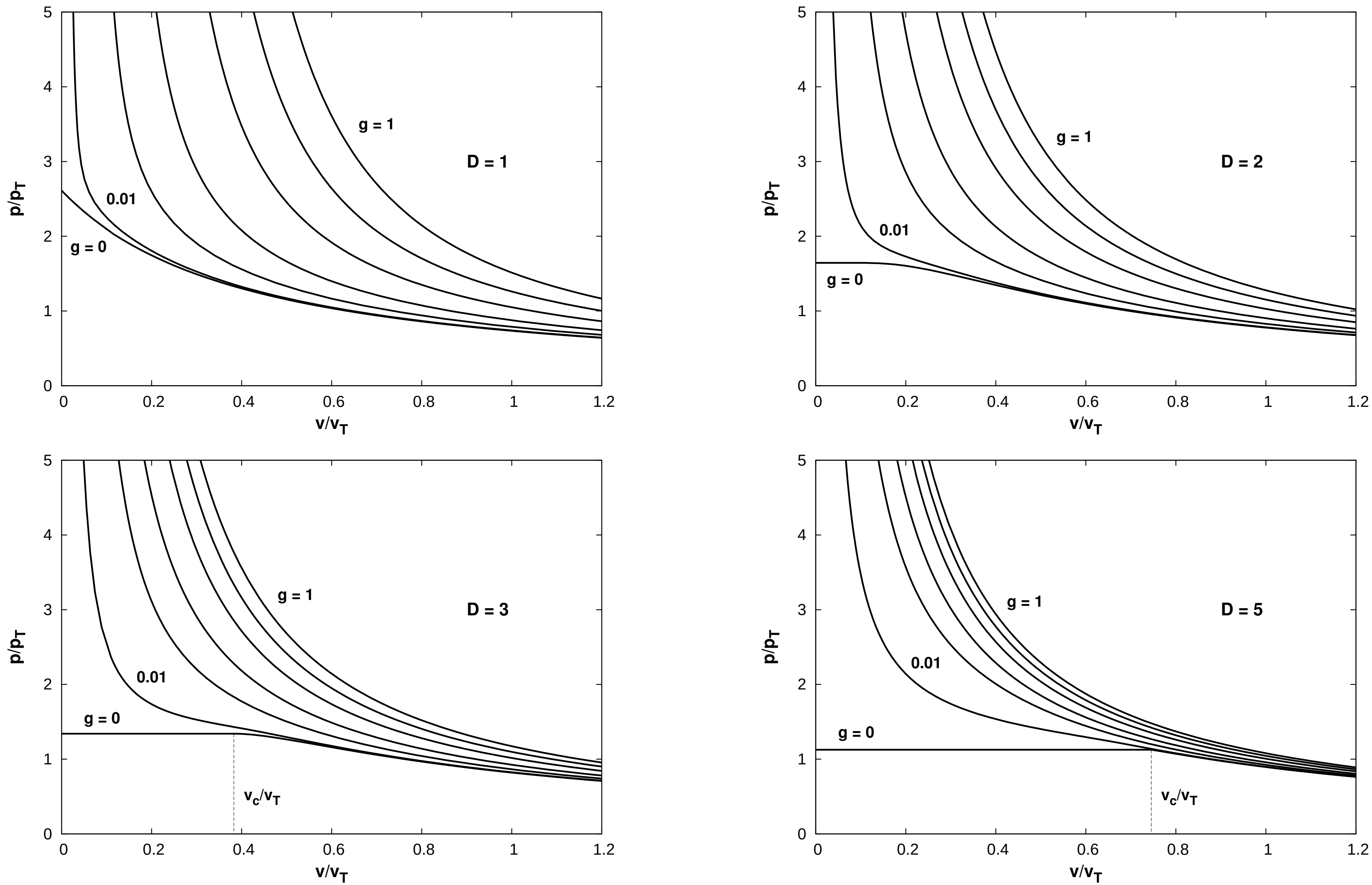
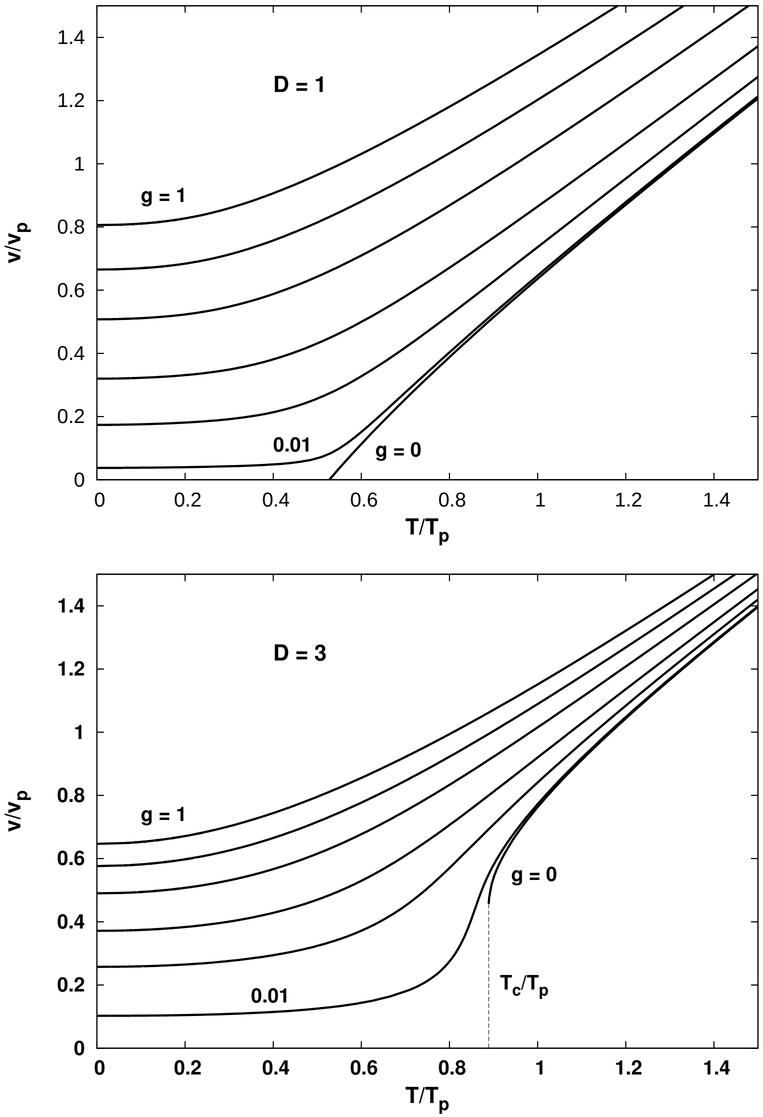
g = 1
g = 0
p/p_v
T/T_v
D = 2
g = 1
g = 0
p/p_v
T/T_v
D = 3
g = 1
0.01
g = 0
T_c/T_v
p/p_v
T/T_v
D = 5
g = 1
0.01
g = 0
T_c/T_v
p/p_v
T/T_v

Isotherms

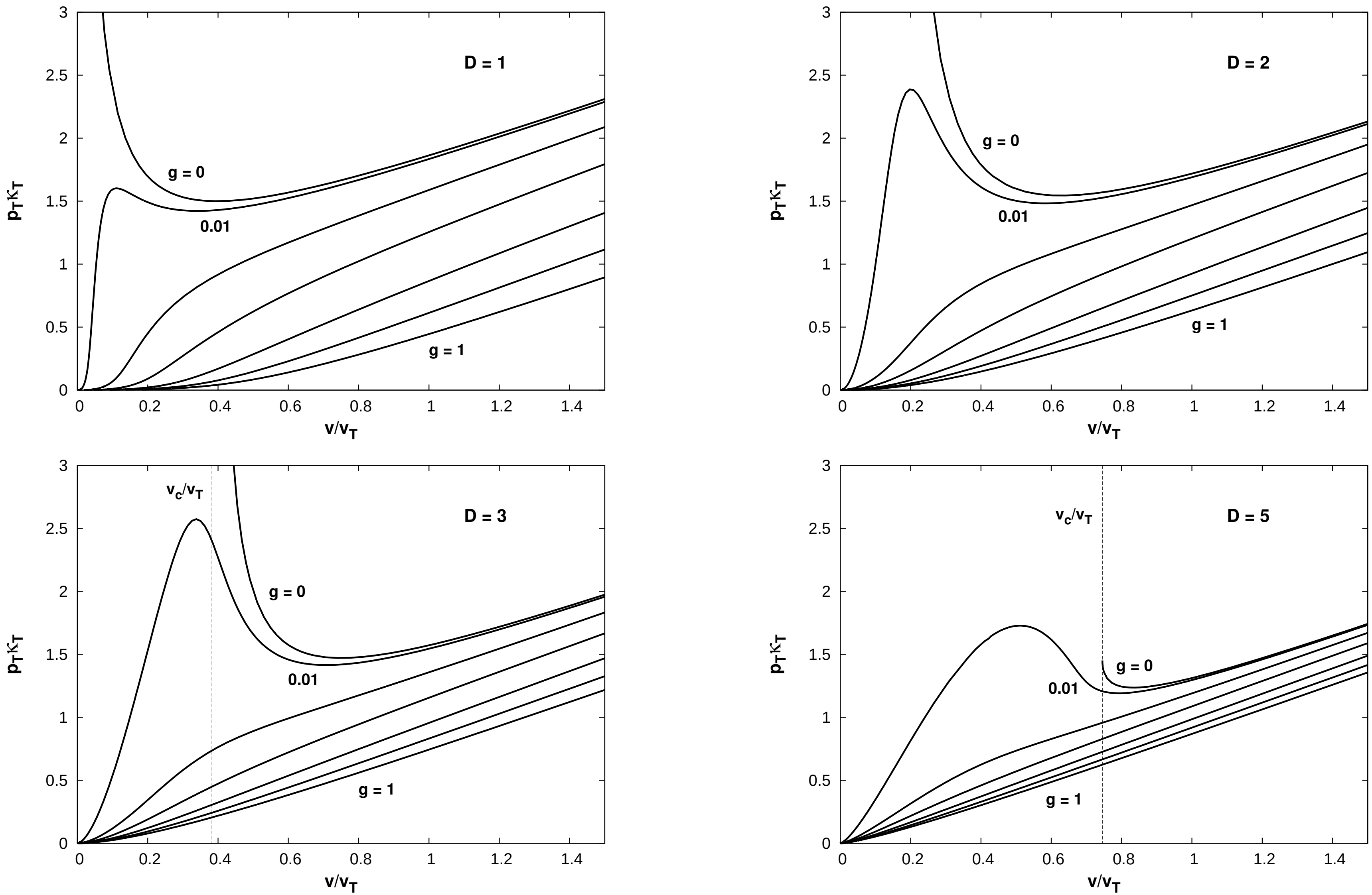


D = 1
g = 1
0.01
g = 0
D = 2
g = 1
0.01
g = 0
D = 3
g = 1
0.01
g = 0
v_c/v_T
D = 5
g = 1
0.01
g = 0
v_c/v_T
p/p_T
v/v_T

Isobars


D = 1
g = 1
0.01
g = 0
v/v_p
T/T_p
D = 2
g = 1
0.01
g = 0
v/v_p
T/T_p
D = 3
g = 1
0.01
g = 0
T_c/T_p
v/v_p
T/T_p
D = 5
g = 1
0.01
g = 0
T_c/T_p
v/v_p
T/T_p

Isothermal Compressibility



Isobaric Expansivity