Domains and Domain Walls in Quantum Spin Chains

Statistical Interaction and Thermodynamics

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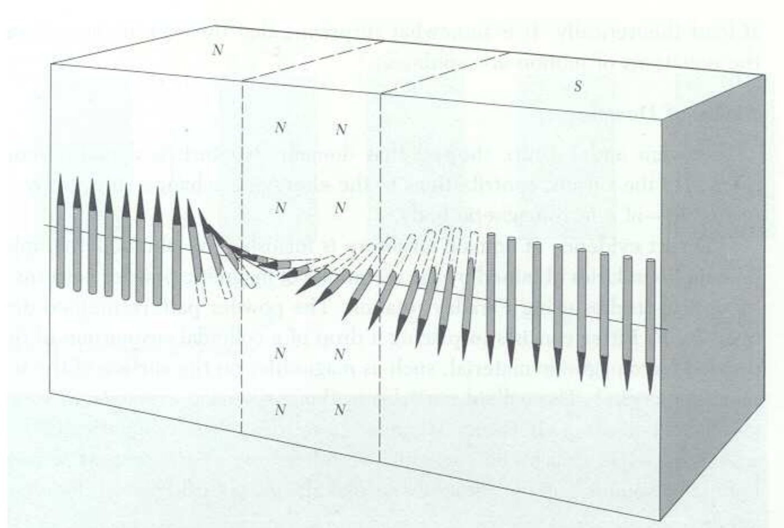
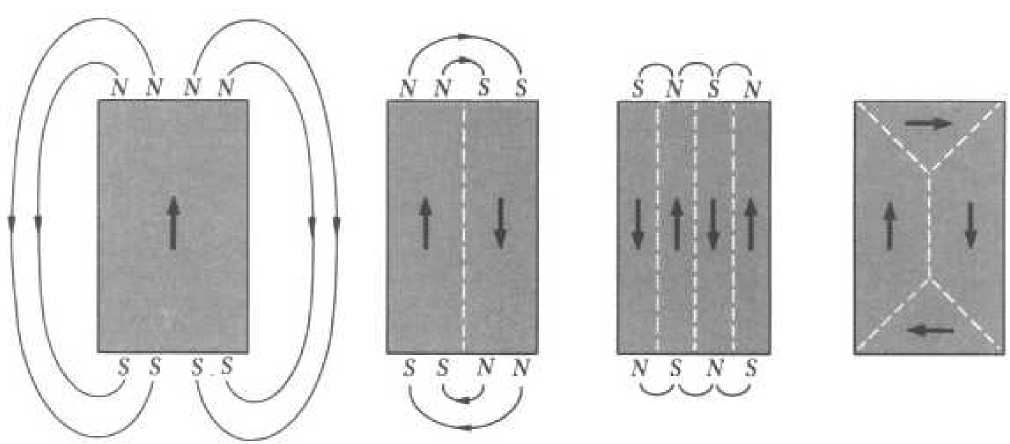
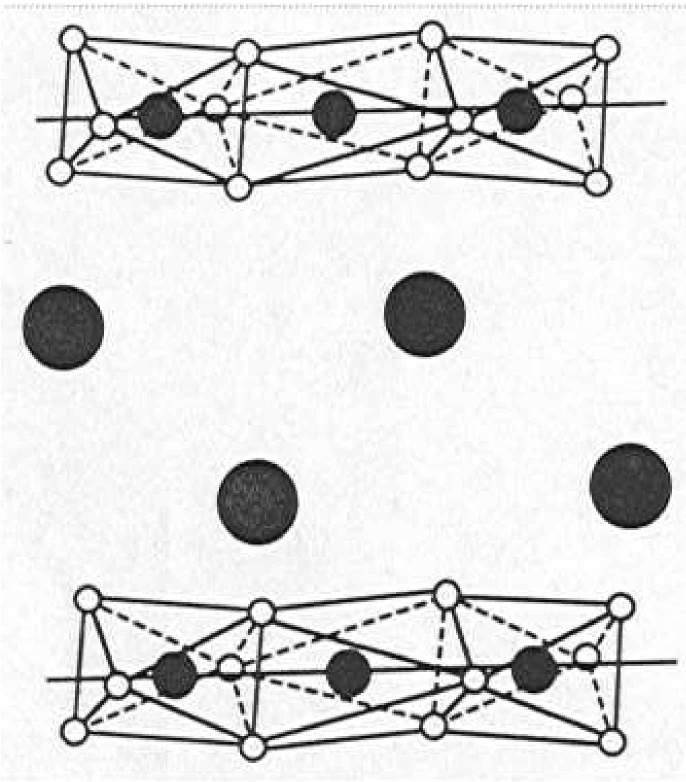
Domains and Domain Walls (Mesoscopic)

- Exchange interaction
  (strong, short-range)
- Magnetic dipolar interaction
  (weak, long range)
- Anisotropy, crystal field



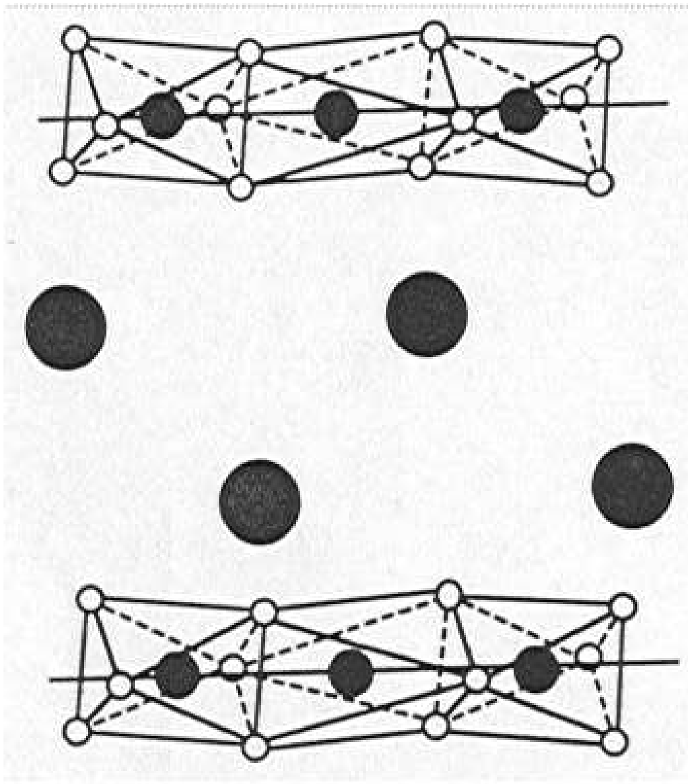


Quantum Spin Chain



$CsCoCl_3$

Quantum Spin Chain



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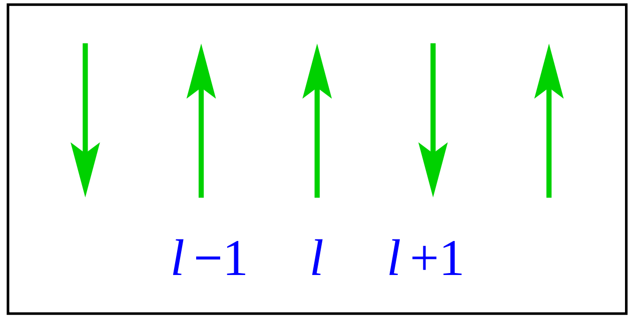

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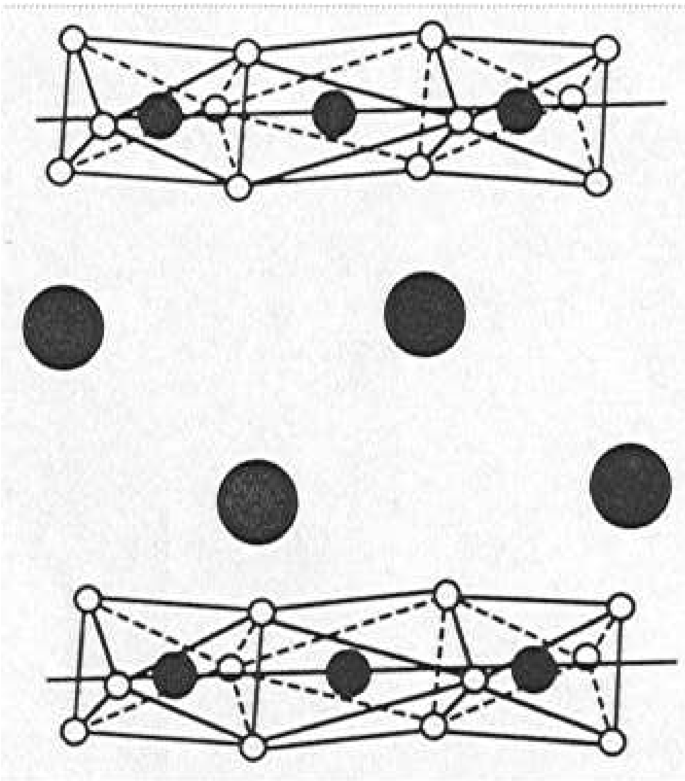
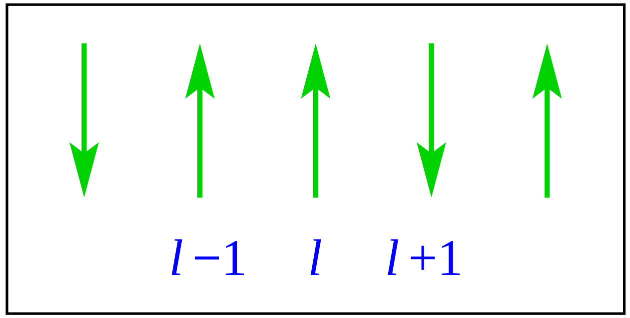
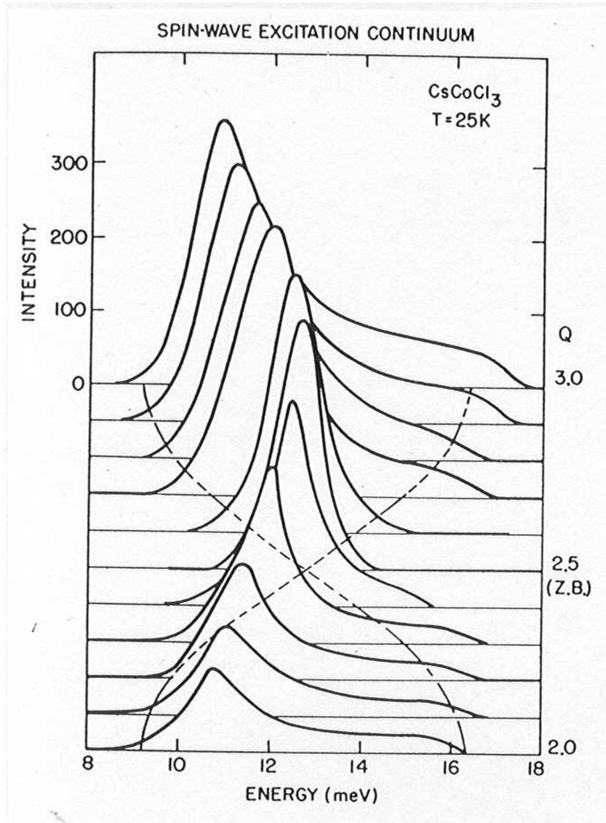




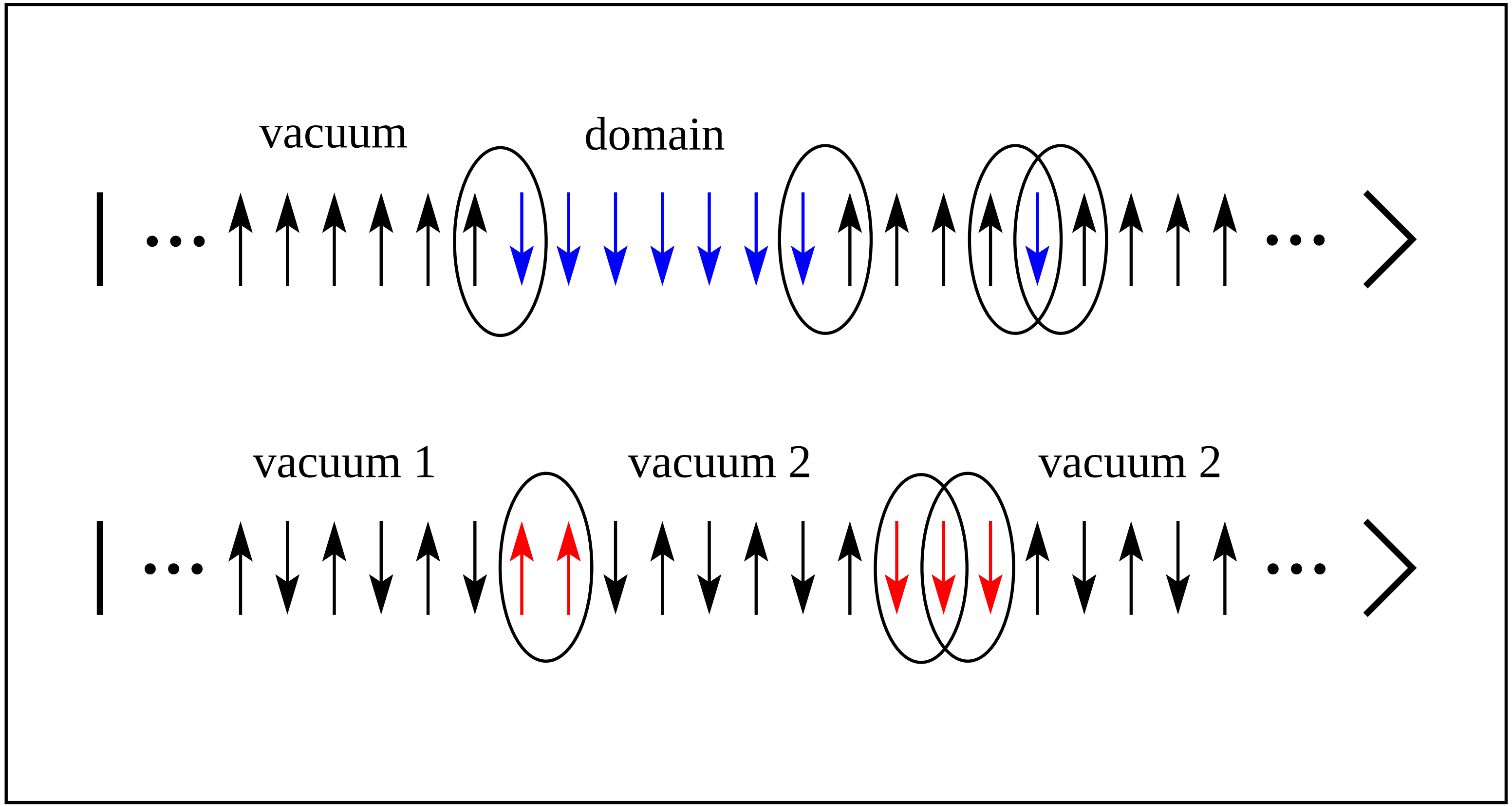


$CsCoCl_3$

[Yoshizawa, Hirakawa, Satija, and Shirane 1981]

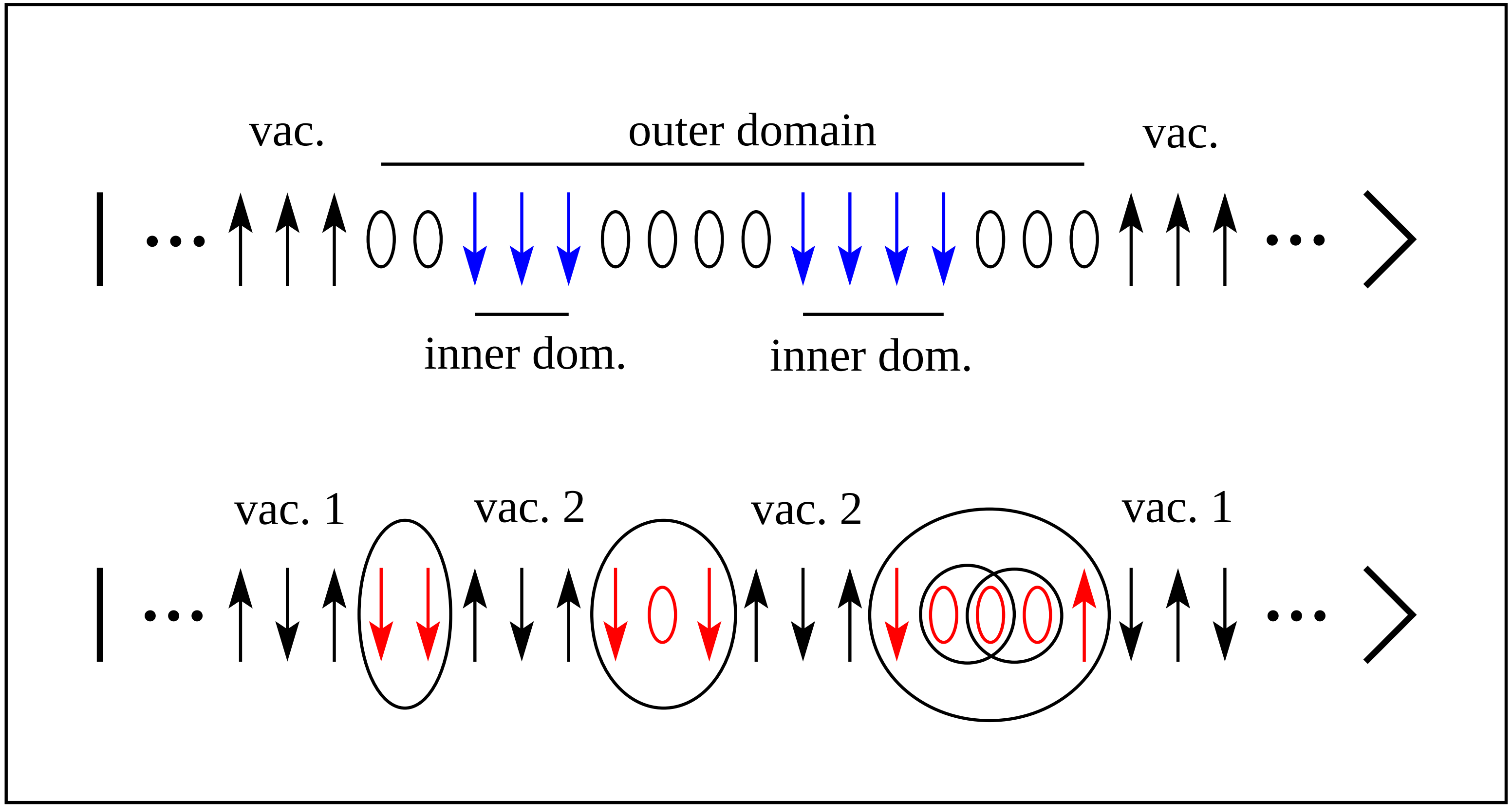
Domains and Domain Walls (Microscopic)

Spin-1/2 Ising chain: $\mathcal{H}_{1/2} = J \sum_{\ell=1}^{N} S_\ell^z S_{\ell+1}^z;$ $\quad S_\ell^z = +\frac{1}{2}, -\frac{1}{2},$ $\quad$ FM $(J < 0)$, AFM $(J > 0)$.



Nested Domains and Composite Domain Walls

Spin-1 Ising chain: $\mathcal{H}_1 = J \sum_{\ell=1}^{N} S_\ell^z S_{\ell+1}^z; \qquad S_\ell^z = +1, 0, -1$



Statistical Interaction

Generalized Pauli principle [Haldane 1991]

How is the number of states accessible to one particle of species i affected if other particles (of any species j) are added?

$$\Delta d_i \doteq -\sum_j g_{ij}\Delta N_j \qquad \Rightarrow \; d_i = A_i - \sum_j g_{ij}(N_j - \delta_{ij}),$$

g_{ij}: coefficients of statistical interaction,
A_i: statistical capacity constants.

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Number of distinct many-particle states with composition $\{N_i\}$:

$$W(\{N_i\}) = \prod_i \begin{pmatrix} d_i + N_i - 1 \\ N_i \end{pmatrix} = \prod_i \frac{\Gamma(d_i + N_i)}{\Gamma(N_i + 1)\Gamma(d_i)}.$$

Packing capacity limited by requirement $d_i > 0$.

Solitons in Spin-1/2 Ising Chain

Soliton content and energy
of nearest-neighbor bonds:

bond	$\uparrow\uparrow$	$\downarrow\downarrow$	$\uparrow\downarrow$	$\downarrow\uparrow$
N_+	1	0	0	0
N_-	0	1	0	0
$\Delta E/J$	$\frac{1}{2}$	$\frac{1}{2}$	0	0

Solitons . . .

- have spin $\pm 1/2$,
- are fundamental,
- are independent,
- are statistically interacting.

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$M_z \backslash N_A$	0	2	4	6	
3	–	–	–	1	1
2	–	–	6	–	6
1	–	9	6	–	15
0	2	12	6	–	20
-1	–	9	6	–	15
-2	–	–	6	–	6
-3	–	–	–	1	1
	2	30	30	2	64

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	2	30	30	2	64

$M_z \backslash N_A$	1	3	5	7	
$7/2$	–	–	–	1	1
$5/2$	–	–	7	–	7
$3/2$	–	14	7	–	21
$1/2$	7	21	7	–	35
$-1/2$	7	21	7	–	35
$-3/2$	–	14	7	–	21
$-5/2$	–	–	7	–	7
$-7/2$	–	–	–	1	1
	14	70	42	2	128

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Statistical Interaction of Solitons

Number of Ising eigenstates with $N_A = N_+ + N_-$ solitons:

- $W_A(N_+, N_-) = \dfrac{2N}{N - N_A} \displaystyle\prod_{\sigma=\pm} \begin{pmatrix} d_\sigma + N_\sigma - 1 \\ N_\sigma \end{pmatrix}$
- $d_\sigma = A_\sigma - \displaystyle\sum_{\sigma'} g_{\sigma\sigma'} (N_{\sigma'} - \delta_{\sigma\sigma'})$
- $A_\sigma = \dfrac{1}{2}(N - 1)$
- $g_{\sigma\sigma'} = \dfrac{1}{2}$
- $M_z = \dfrac{1}{2}(N_+ - N_-)$
- $\epsilon_\pm = \dfrac{1}{2}(J \pm h) \quad (\mathrm{AFM})$
- $\displaystyle\sum_{N_+, N_-} W_A(N_+, N_-) = 2^N$

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particle	n_+	n_-
motif	$\uparrow\uparrow$	$\downarrow\downarrow$
$N_+ + N_-$	$1+0$	$0+1$
A_σ	$\frac{N-1}{2}$	$\frac{N-1}{2}$
ϵ_σ	$\frac{1}{2}(J+h)$	$\frac{1}{2}(J-h)$

$g_{\sigma\sigma'}$	$+$	$-$
$+$	$\frac{1}{2}$	$\frac{1}{2}$
$-$	$\frac{1}{2}$	$\frac{1}{2}$

Thermodynamics with Statistical Interaction

Specifications of quantum many-body system:

- particle energies ϵ_i,
- statistical interaction coefficients g_{ij},
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Two tasks:

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- extremum problem: $\delta(U - TS - \mu\mathcal{N}) = 0$.

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Grand potential [Wu 1994]: $\Omega = -k_B T \ln Z, \quad Z = \prod_i \left(\frac{1+w_i}{w_i}\right)^{A_i}$,

$$\frac{\epsilon_i - \mu}{k_B T} = \ln(1+w_i) - \sum_j g_{ji} \ln\left(\frac{1+w_i}{w_i}\right).$$

Thermodynamics of Solitons

Specifications: $\epsilon_\pm = \frac{1}{2}(J \pm h), \quad g_{\sigma\sigma'} = \frac{1}{2}, \quad A_\pm = \frac{1}{2}(N-1).$

Grand partition function: $Z = \left(\frac{1+w_+}{w_+}\right)^{N/2} \left(\frac{1+w_-}{w_-}\right)^{N/2},$

$$\frac{J \pm h}{2k_B T} = \ln(1+w_\pm) - \frac{1}{2}\ln\frac{1+w_\pm}{w_\pm} - \frac{1}{2}\ln\frac{1+w_\mp}{w_\mp}.$$

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Solution:

$$w_\pm = \frac{1}{2}\left[e^{\pm h/k_BT} - 1 + \sqrt{(e^{\pm h/k_BT}-1)^2 + 4e^{(J\pm h)/k_BT}}\right],$$

$$Z = \left[e^{2K}\left(\cosh H + \sqrt{\sinh^2 H + e^{-4K}}\right)\right]^N, \quad K \doteq -\frac{J}{4k_BT}, \quad H \doteq -\frac{h}{2k_BT}.$$

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Comparison with transfer matrix solution:

$$Z = Z_N\, e^{NK}.$$

Transfer Matrix Solution

Ising chain with spin $s = 1/2$ in a magnetic field:

$$\mathcal{H} = \sum_{n=1}^{N} \left[J S_n^z S_{n+1}^z + \frac{1}{2} h \left(S_n^z + S_{n+1}^z \right) \right].$$

Canonical partition function (use $\sigma_n = 2S_n^z = \pm 1$):

$$Z_N = \sum_{\sigma_1,\ldots,\sigma_N} \exp\left(\sum_{n=1}^{N} \left[K\sigma_n\sigma_{n+1} + \frac{1}{2} H(\sigma_n + \sigma_{n+1}) \right] \right) = \mathrm{Tr}\left[\mathbf{V}^N \right] = \lambda_+^N \left[1 + \left(\frac{\lambda_-}{\lambda_+} \right)^N \right].$$

Transfer matrix [Kramers and Wannier 1941]:

$$\mathbf{V} = \begin{pmatrix} e^{K+H} & e^{-K} \\ e^{-K} & e^{K-H} \end{pmatrix}, \quad V(\sigma_n, \sigma_{n+1}) = \exp\left(K\sigma_n\sigma_{n+1} + \frac{1}{2} H(\sigma_n + \sigma_{n+1}) \right)$$

Eigenvalues: $\lambda_\pm = e^K \left[\cosh H \pm \sqrt{\sinh^2 H + e^{-4K}} \right], \quad K \doteq -\frac{J}{4k_B T}, \quad H \doteq -\frac{h}{2k_B T}.$

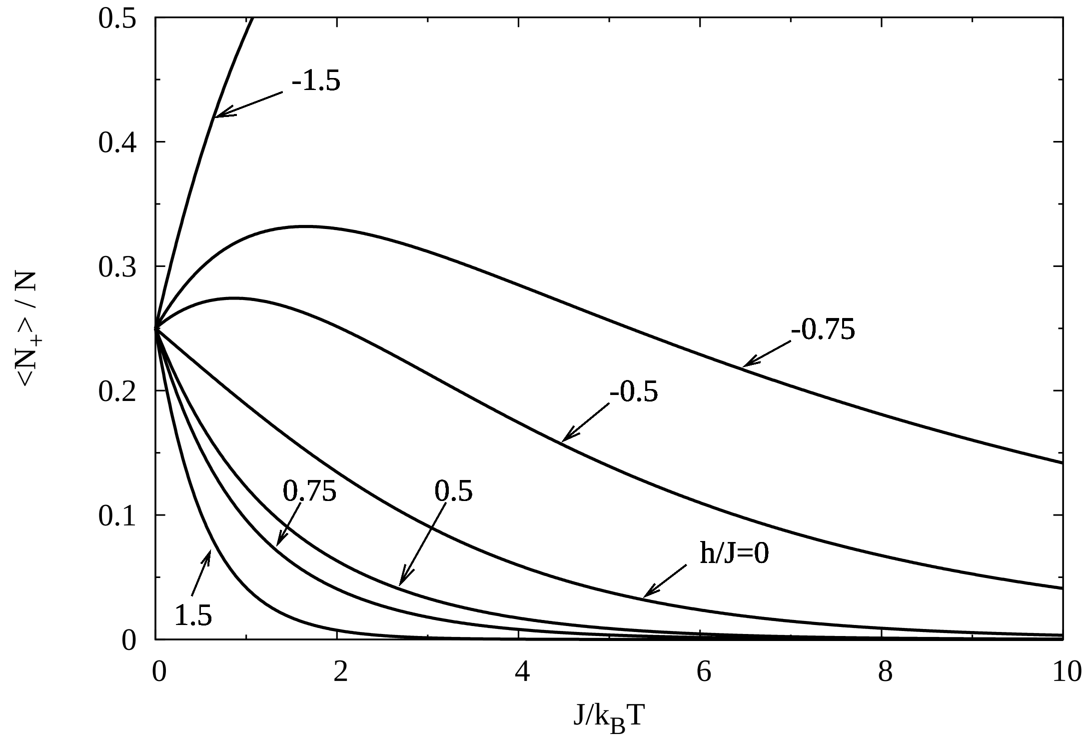
Soliton Occupancies

Linear relations [Wu 1994]: $w_i \langle N_i \rangle + \sum_j g_{ij} \langle N_j \rangle = A_i, \quad i = 1, 2, \ldots.$

Solitons in $s = 1/2$ Ising chain at $h \neq 0$:

$$\left(w_\pm + \frac{1}{2} \right) \langle N_\pm \rangle + \frac{1}{2} \langle N_\mp \rangle = \frac{N}{2}$$

$$\Rightarrow \langle N_\pm \rangle = \frac{N w_\mp}{2 w_+ w_- + w_+ + w_-}$$



$$\frac{\langle N_\pm \rangle}{N} = \frac{1}{2} \frac{e^{\pm H} \left[\sqrt{\sinh^2 H + e^{-4K}} \pm \sinh H \right]}{\sinh^2 H + e^{-4K} + \cosh H \sqrt{\sinh^2 H + e^{-4K}}} \xrightarrow{h \to 0} \frac{1/2}{e^{-2K} + 1}$$

Solitons in Spin-1 Ising Chain

Soliton content and energy of nearest-neighbor bonds:

bond	$\uparrow\uparrow$	$\circ\circ$	$\downarrow\downarrow$	$\uparrow\circ$	$\circ\uparrow$	$\downarrow\circ$	$\circ\downarrow$	$\uparrow\downarrow$	$\downarrow\uparrow$
N_+	2	1	0	1	1	0	0	0	0
N_-	0	1	2	0	0	1	1	0	0
$\Delta E/J$	2	1	2	1	1	1	1	0	0

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N_+	2	1	0	1	1	0	0	0	0
N_-	0	1	2	0	0	1	1	0	0
$\Delta E/J$	2	1	2	1	1	1	1	0	0

$M_z\backslash N_A$	0	2	4	6	
3	–	–	–	1	1
2	–	–	3	–	3
1	–	3	3	–	6
0	–	6	–	1	7
-1	–	3	3	–	6
-2	–	–	3	–	3
-3	–	–	–	1	1
	0	12	12	3	27

$M_z\backslash N_A$	0	2	4	6	8	
4	–	–	–	–	1	1
3	–	–	–	4	–	4
2	–	–	6	4	–	10
1	–	4	8	4	–	16
0	2	–	16	–	1	19
-1	–	4	8	4	–	16
-2	–	–	6	4	–	10
-3	–	–	–	4	–	4
-4	–	–	–	–	1	1
	2	8	48	20	3	81

Soliton Pairs in Spin-1 Ising Chain

particle	r_+	r_-	r_0	q_+	q_-	q_0
motif	$\uparrow\uparrow$	$\downarrow\downarrow$	$\circ\circ$	$\uparrow\circ\uparrow$	$\downarrow\circ\downarrow$	$\uparrow\circ\downarrow, \downarrow\circ\uparrow$
$N_+ + N_-$	$2+0$	$0+2$	$1+1$	$2+0$	$0+2$	$1+1$
ϵ_m	$2J$	$2J$	J	$2J$	$2J$	$2J$

Independent particles ...

- are composite (soliton pairs),
- have spin $+1, 0, -1$,
- come in six species:
  - confined soliton pairs (r_+, r_-),
  - deconfined soliton pairs (q_+, q_0, q_-),
  - spacer particle (r_0).

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particle	r_+	r_-	r_0	q_+	q_-	q_0
motif	$\uparrow\uparrow$	$\downarrow\downarrow$	$\circ\circ$	$\uparrow\circ\uparrow$	$\downarrow\circ\downarrow$	$\uparrow\circ\downarrow, \downarrow\circ\uparrow$
$N_+ + N_-$	$2+0$	$0+2$	$1+1$	$2+0$	$0+2$	$1+1$
ϵ_m	$2J$	$2J$	J	$2J$	$2J$	$2J$

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  - spacer particle (r_0).

Sample configurations:

- $\downarrow\downarrow\downarrow\uparrow\uparrow$: $2r_- + r_+$,
- $\uparrow\circ\circ\circ\uparrow$: $q_+ + 2r_0$
- $\downarrow\circ\uparrow\circ\circ\downarrow\downarrow$: $2q_0 + r_0 + r_-$

Statistical Interaction of Soliton Pairs

Multiplicity expression for soliton pairs:

$$W_6(\{X_m\}) = \frac{2N}{N - N_\Sigma} \prod_{m=1}^{6} \begin{pmatrix} d_m + X_m - 1 \\ X_m \end{pmatrix}$$

$$d_m = A_m - \sum_{m'} g_{mm'}(X_{m'} - \delta_{mm'})$$

$$N_\Sigma = X_1 + X_2 + X_3 + 2(X_4 + X_5 + X_6)$$

$g_{mm'}$	1	2	3	4	5	6
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$
3	0	0	0	-1	-1	-1
4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
5	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
6	1	1	1	2	2	2

particle	r_+	r_-	r_0	q_+	q_-	q_0
motif	$\uparrow\uparrow$	$\downarrow\downarrow$	$\circ\circ$	$\uparrow\circ\uparrow$	$\downarrow\circ\downarrow$	$\uparrow\circ\downarrow, \downarrow\circ\uparrow$
$N_+ + N_-$	$2+0$	$0+2$	$1+1$	$2+0$	$0+2$	$1+1$
m	1	2	3	4	5	6
A_m	$\frac{N-1}{2}$	$\frac{N-1}{2}$	0	$\frac{N}{2} - 1$	$\frac{N}{2} - 1$	$N - 2$
ϵ_m	$2J$	$2J$	J	$2J$	$2J$	$2J$

Thermodynamics of Soliton Pairs

Six coupled equations for given ϵ_i, g_{ij}, and $\mu = 0$:

$$\frac{\epsilon_i - \mu}{k_B T} = \ln(1 + w_i) - \sum_j g_{ji} \ln\left(\frac{1 + w_j}{w_j}\right), \quad i = 1, , \ldots, 6.$$

Set $K \doteq -J/k_B T$.

$$w_1 = w_2, \quad w_4 = w_5, \quad \frac{1 + w_3}{1 + w_1} = e^K, \quad \frac{w_3}{w_6} = e^K, \quad w_4 = 1 + w_6, \quad \frac{2 + w_6}{w_1 w_6} = e^{2K}.$$

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Solution: $w_3 = \cosh K - \frac{1}{2} + \sqrt{\left(\cosh K - \frac{1}{2}\right)^2 + 2}.$

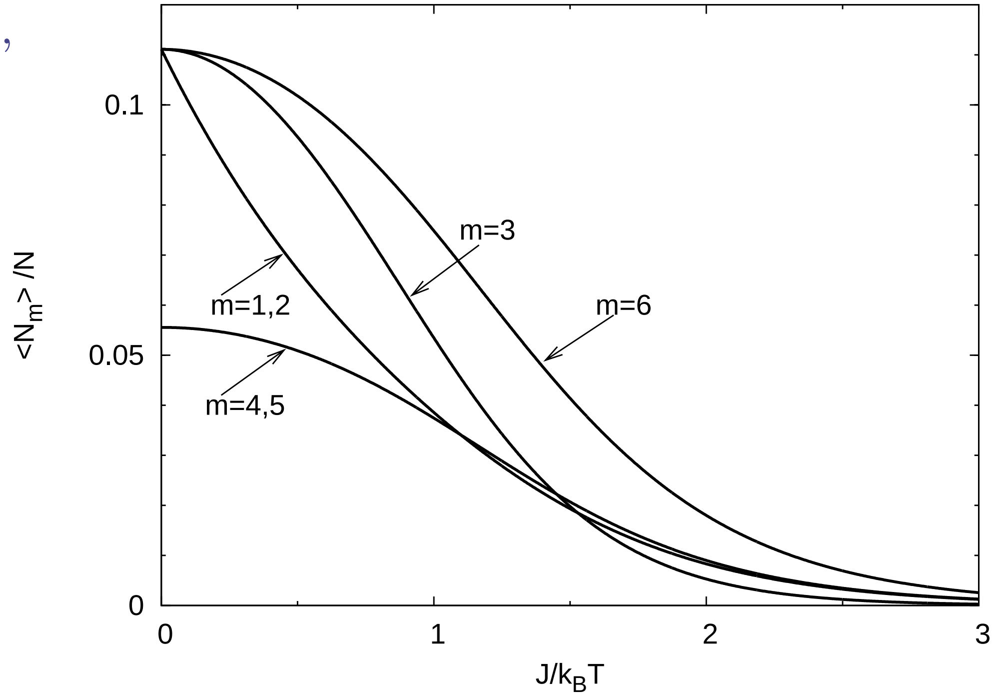
Grand partition function: $Z = \prod_{i=1}^{6} \left[\frac{1 + w_i}{w_i}\right]^{A_i} = \left[(1 + w_3)e^K\right]^N = Z_N e^{NK}.$

Soliton-Pair Occupancies at $h = 0$

Linear relations: $w_i \langle N_i \rangle + \sum_j g_{ij} \langle N_j \rangle = A_i, \quad i = 1, \ldots, 6.$

Symmetry implies $\langle N_1 \rangle = \langle N_2 \rangle, \quad \langle N_4 \rangle = \langle N_5 \rangle = \frac{1}{2} \langle N_6 \rangle.$

- $\langle N_1 \rangle = \langle N_2 \rangle = \frac{N}{2} \frac{w_3(w_3^2 + 2e^K)}{(w_3+1)(w_3^2 e^{-K} + 4w_3 + 2 - 2e^K)},$
- $\langle N_3 \rangle = \frac{2N(w_3 + 1 - e^K)}{(w_3+1)(w_3^2 e^{-K} + 4w_3 + 2 - 2e^K)},$
- $\langle N_4 \rangle = \langle N_5 \rangle = \frac{1}{4} w_3 \langle N_3 \rangle,$
- $\langle N_6 \rangle = \frac{1}{2} w_3 \langle N_3 \rangle.$



Combinatorics of Domains

Number of Ising states with $\{N_1, N_2, \ldots\}$ strings:

- $W_\mu(\{N_\mu\}) = \dfrac{N}{N-r} \prod_\mu \begin{pmatrix} d_\mu + N_\mu - 1 \\ N_\mu \end{pmatrix}$
- $d_\mu = A_\mu - \sum_{\mu'} g_{\mu\mu'}(N_{\mu'} - \delta_{\mu\mu'})$
- $A_\mu = N - \mu$
- $g_{\mu\mu'} = \begin{cases} \mu', & \mu < \mu' \\ \mu' + 1, & \mu \geq \mu' \end{cases}$
- $M_z = \dfrac{N}{2} - r, \quad r = \sum_\mu \mu N_\mu$
- $\epsilon_\mu = |J| + \mu h \quad \text{(FM)}$
- $\sum_{\{N_\mu\}} W_\mu(\{N_\mu\}) = 2^N.$

$\{\mid\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\rangle\}_1^{+3}$	$-$	1
$\{\mid\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\rangle\}_6^{+2}$	$N_1 = 1$	6
$\{\mid\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\rangle\}_6^{+1}$	$N_2 = 1$	
$\{\mid\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow\rangle\}_6^{+1}$	$N_1 = 2$	
$\{\mid\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow\rangle\}_3^{+1}$	$N_1 = 2$	15
$\{\mid\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\rangle\}_6^{0}$	$N_3 = 1$	
$\{\mid\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\rangle\}_6^{0}$	$N_1 = N_2 = 1$	
$\{\mid\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle\}_2^{0}$	$N_1 = 3$	
$\{\mid\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\rangle\}_6^{0}$	$N_1 = N_2 = 1$	20
$\{\mid\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\rangle\}_3^{-1}$	$N_2 = 2$	
$\{\mid\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\rangle\}_6^{11}$	$N_1 = N_3 = 1$	
$\{\mid\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\rangle\}_6^{-1}$	$N_4 = 1$	15
$\{\mid\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle\}_6^{-2}$	$N_5 = 1$	6
$\{\mid\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\rangle\}_1^{-3}$	$N_6 = 1$	1

Nested Domains in Spin-1 Ising Chain

Nested two-level systems:

- μ-string vacuum: $\vert\uparrow\uparrow\cdots\uparrow\rangle$ (FM ground state),
- ν-string vacuum: $\vert\cdots\uparrow\underline{\circ\circ\cdots\circ}\uparrow\cdots\rangle$ (μ-string particle).

1	$\{\vert\uparrow\uparrow\uparrow\uparrow\rangle\}_1^{+4}$	$\{\vert\uparrow\uparrow\uparrow\uparrow\rangle\}_{1\times1}^{+4}$,	1
4	$\{\vert\uparrow\uparrow\uparrow\underline{\circ}\rangle\}_4^{+3}$	$\{\vert\uparrow\uparrow\uparrow\circ\rangle\}_{4\times1}^{+3}$, $\{\vert\uparrow\uparrow\uparrow\downarrow\rangle\}_{4\times1}^{+2}$,	8
4	$\{\vert\uparrow\uparrow\underline{\circ\circ}\rangle\}_4^{+2}$	$\{\vert\uparrow\uparrow\circ\circ\rangle\}_{4\times1}^{+2}$, $\{\vert\uparrow\uparrow\downarrow\circ\rangle\}_{4\times2}^{+1}$, $\{\vert\uparrow\uparrow\downarrow\downarrow\rangle\}_{4\times1}^{0}$,	16
4	$\{\vert\uparrow\underline{\circ\circ\circ}\rangle\}_4^{+1}$	$\{\vert\uparrow\circ\circ\circ\rangle\}_{4\times1}^{+1}$, $\{\vert\uparrow\downarrow\circ\circ\rangle\}_{4\times3}^{0}$, $\{\vert\uparrow\downarrow\downarrow\circ\rangle\}_{4\times3}^{-1}$, $\{\vert\uparrow\downarrow\downarrow\downarrow\rangle\}_{4\times1}^{-2}$	32
1	$\{\vert\underline{\circ\circ\circ\circ}\rangle\}_1^{0}$	$\{\vert\circ\circ\circ\circ\rangle\}_{1\times1}^{0}$, $\{\vert\downarrow\circ\circ\circ\rangle\}_{1\times4}^{-1}$ $\{\vert\downarrow\downarrow\circ\circ\rangle\}_{1\times4}^{-2}$, $\{\vert\downarrow\circ\downarrow\circ\rangle\}_{1\times2}^{-2}$, $\{\vert\downarrow\downarrow\downarrow\circ\rangle\}_{1\times4}^{-3}$, $\{\vert\downarrow\downarrow\downarrow\downarrow\rangle\}_{1\times1}^{-4}$,	16
2	$\{\vert\uparrow\underline{\circ}\uparrow\underline{\circ}\rangle\}_2^{+2}$	$\{\vert\uparrow\circ\uparrow\circ\rangle\}_{2\times1}^{+2}$, $\{\vert\uparrow\downarrow\uparrow\circ\rangle\}_{4\times1}^{+1}$, $\{\vert\uparrow\downarrow\uparrow\downarrow\rangle\}_{2\times1}^{0}$,	8
16			81

Link to nested coordinate Bethe ansatz.

Combinatorics of Nested Domains

Multiplicity expression:

$$
\begin{aligned}
W\left(\{N_\mu\},\{N_\nu^{(\mu)}\}\right) &= \frac{N}{N-r}\prod_\mu \left(\begin{array}{c} d_\mu + N_\mu - 1 \\ N_\mu \end{array}\right) \\
&\times \frac{\mu}{\mu - r_\mu}\prod_\nu \left(\begin{array}{c} d_\nu^{(\mu)} + N_\nu^{(\mu)} - 1 \\ N_\nu^{(\mu)} \end{array}\right),
\end{aligned}
$$

$$
d_\mu = N - \mu - \sum_{\mu'} g_{\mu\mu'}(N_{\mu'} - \delta_{\mu\mu'}), \quad d_\nu^{(\mu)} = \mu - \nu - \sum_{\nu'} g_{\nu\nu'}(N_{\nu'}^{(\mu)} - \delta_{\nu\nu'}),
$$

$$
r = \sum_\mu \mu N_\mu, \quad r_\mu = \sum_\nu \nu N_\nu^{(\mu)}, \qquad g_{\mu\mu'} = \left\{ \begin{array}{ll} \mu', & \mu < \mu', \\ \mu' + 1 & \mu \geq \mu' \end{array} \right. .
$$

Link to nested thermodynamic Bethe ansatz.

Statistical Mechanics of Domains I

Strings of restricted length: $\mu = 1, 2, \ldots, M \ll N$.

Grand potential: $\Omega_M(T) = -k_B T \sum_{\mu=1}^{M} A_\mu \ln\left(\frac{w_\mu + 1}{w_\mu}\right)$.

$$4K_\mu = \ln(w_\mu + 1) - \sum_{\mu'=1}^{M} g_{\mu'\mu} \ln \frac{w_{\mu'} + 1}{w_{\mu'}}, \quad \mu = 1, \ldots, M.$$

- $A_\mu = N - \mu$,
- $g_{\mu\mu'} = \begin{cases} \mu', & \mu < \mu' \\ \mu' + 1, & \mu \geq \mu' \end{cases}$,
- $K_\mu = \frac{\epsilon_\mu}{k_B T}$.

Spin-1/2 ising chain: $\epsilon_\mu = |J| + \mu h$.

Thermodynamic limit: first $N \to \infty$ then $M \to \infty$.

Statistical Mechanics of Domains II

Solution for $\epsilon_\mu = |J|$, finite M, and $N \to \infty$.

Introduce $\xi_\mu \doteq \ln(w_\mu + 1)$, $q \doteq e^{-4K}$, $K = -|J|/4k_BT$.

Introduce $\Phi_M \doteq -\frac{1}{4K}\sum_{\mu=1}^{M} \ln\left(1 - e^{-\xi_\mu}\right)$.

Grand potential per site: $\omega_M(T) \doteq \lim_{N\to\infty} \frac{1}{N}\Omega_M(T) = -|J|\Phi_M$.

Polynomial equation: $qx^{(M+2)} + (1-q)x^2 - 2x + 1 = 0, \quad x \doteq q^{\Phi_M} < 1$.

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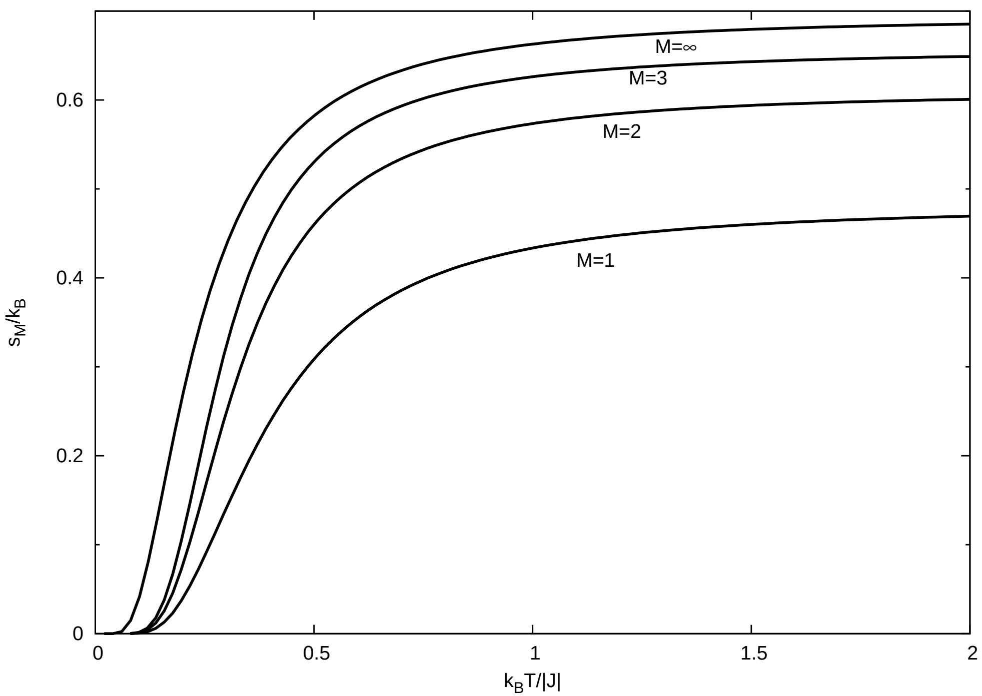
Solution for spin-1/2 Ising chain $(M \to \infty)$: $x = \dfrac{1}{1+\sqrt{q}}$.

$$\Rightarrow\ \omega_\infty(T) = -k_BT\ln\left(1 + e^{-|J|/2k_BT}\right).$$

Entropy of Domains

Entropy per site on infinite lattice: $s_M(T) \doteq \lim_{N\to\infty} N^{-1} S_M(T) = -\frac{d}{dT}\omega_M(T)$.

- $\frac{s_1(K)}{k_B} = \ln\left(\frac{\sqrt{1+4e^{-4K}}+1}{2}\right) + \frac{8Ke^{-4K}}{1+4e^{-4K}+\sqrt{1+4e^{-4K}}}$ (1-strings only),
- $\frac{s_\infty(K)}{k_B} = \ln\left(1+e^{-2K}\right) + \frac{2K}{e^{2K}+1}$ (all strings allowed).



Distribution of Domains

Normalized distribution: $\langle \hat{n}_\mu \rangle = \langle n_\mu \rangle \left(\sum_{\mu'=1}^{M} \langle n_{\mu'} \rangle \right)^{-1}, \qquad \langle n_\mu \rangle \doteq \lim_{N\to\infty} \frac{\langle N_\mu \rangle}{N}.$

Solve the linear equations $w_\mu \langle n_\mu \rangle + \sum_{\mu'=1}^{M} \mu' \langle n_{\mu'} \rangle + \sum_{\mu'=1}^{\mu} \langle n_{\mu'} \rangle = 1, \quad \mu = 1, \ldots, M.$

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Consider the case $M \to \infty$: $w_\mu \langle n_\mu \rangle + \sum_{\nu=1}^{\mu} \langle n_\nu \rangle = \zeta, \quad \mu = 1, 2, \ldots$

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Result: $\langle \hat{n}_\mu \rangle = \frac{\sqrt{q}}{(1+\sqrt{q})^\mu} = \frac{e^{-2K}}{(1+e^{-2K})^\mu}, \quad \mu = 1, 2, \ldots$ [Denisov and Hänggi 2005]

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Pascal's distribution: $P(\mu) = \gamma(1-\gamma)^{\mu-1}, \qquad \gamma = \frac{e^{-K}}{e^K + e^{-K}}$.

Open Questions

- Generalization to spin-3/2 Ising chain:
  - independent particles contain 2,3,...,6 solitons,
  - confined soliton compounds (e.g. $\Uparrow\Uparrow$, $\Downarrow\Downarrow$),
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- Links to coordinate Bethe ansatz and thermodynamic Bethe ansatz (including nesting).