Self-Gravitating Lattice Gas

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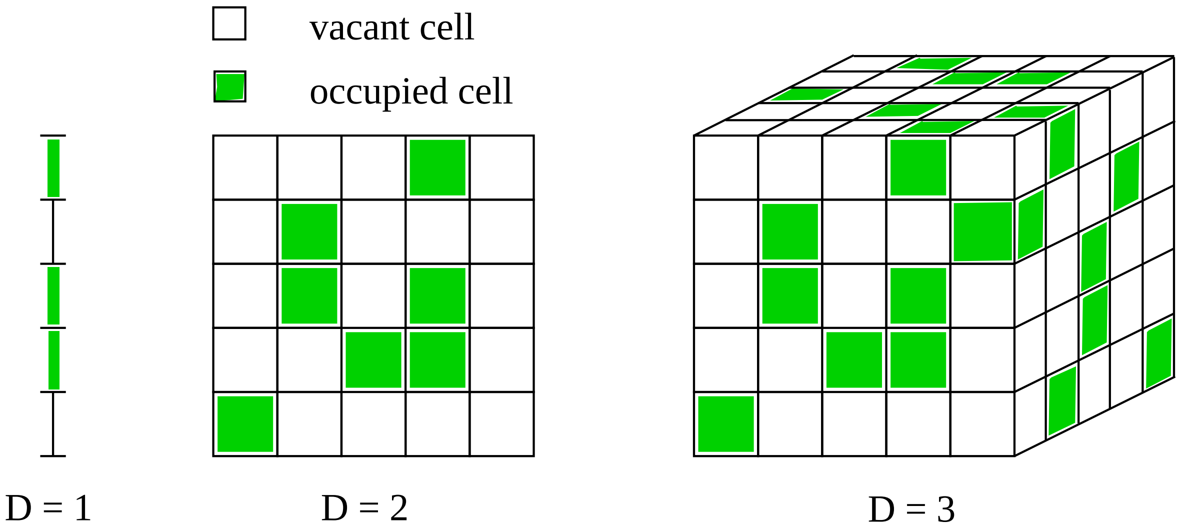
Ideal Lattice Gas

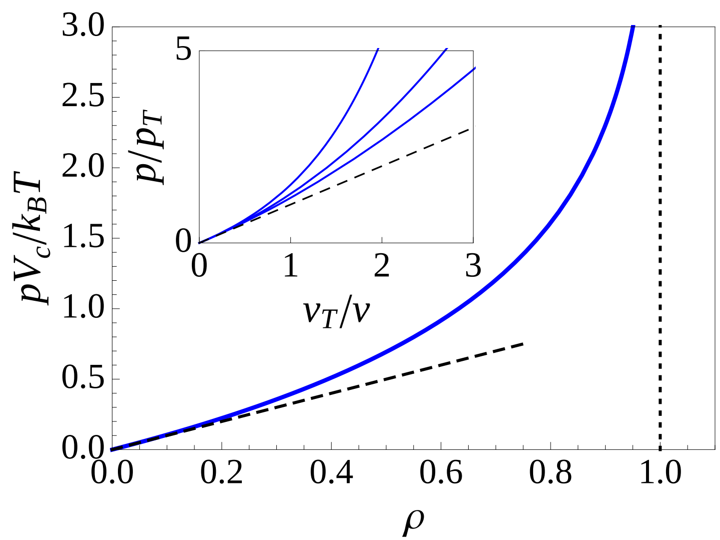
- volume of cell: V_c
- number of cells: N_c
- number of particles: N
- density: $\rho = N/N_c$



Equations of state (EOS):

- Lattice gas: $\dfrac{pV_c}{k_B T} = -\ln(1-\rho)$
- FD gas: $\dfrac{p}{P_T} = f_{\mathcal{D}/2+1}(z), \quad \rho \doteq \dfrac{v_T}{v} = f_{\mathcal{D}/2}(z)$

$p \sim \rho^{(\mathcal{D}+2)/\mathcal{D}}$ (isotherm)



Equilibrium Condition

Heterogeneous environment:

- Presence of external potential $\mathcal{U}(\mathbf{r})$.
- Equilibrium condition has two parts:
  - *equation of state* (EOS):
    (local) thermal equilibrium from minimized free energy density,
  - *equation of motion* (EOM):
    mechanical equilibrium from balanced forces.
- Equilibrium establishes density profile $\rho(\mathbf{r})$ and pressure profile $p(\mathbf{r})$.
- Temperature T remains uniform.
- Potential $\mathcal{U}(\mathbf{r})$ here represented by long-range gravitational interactions.
- Mean-field approach proven exact.
- Ensemble inequivalence to be heeded.

Gravity in $\mathcal{D} = 1, 2, 3$ Dimensions

Interaction potential $\mathcal{V}_{ij}$ and force F_{ij} between occupied cells of mass m_c at distance r_{ij}:

$$\frac{\mathcal{V}_{ij}}{Gm_\mathrm{c}^2}\begin{cases} r_{ij} & : \mathcal{D}=1, \\ \ln r_{ij} & : \mathcal{D}=2, \\ -r_{ij}^{-1} & : \mathcal{D}=3. \end{cases} \qquad F_{ij} = -\frac{Gm_\mathrm{c}^2}{r_{ij}^{\mathcal{D}-1}}.$$

Gauss' law: $\oint d\mathbf{A}\cdot\mathbf{g}(\mathbf{r}) = -\mathcal{A}_\mathcal{D} G m_\mathrm{in}, \quad \mathcal{A}_\mathcal{D} = \frac{2\pi^{\mathcal{D}/2}}{\Gamma(\mathcal{D}/2)} = \begin{cases} 2 & : \mathcal{D}=1, \\ 2\pi & : \mathcal{D}=2, \\ 4\pi & : \mathcal{D}=3. \end{cases}$

Symmetric self-gravitating cluster with center of mass at $\mathbf{r}=0$.

Gravitational field: $g(r) = -\dfrac{m_\mathrm{in}}{r^{\mathcal{D}-1}}$.

Gravitational potential $\mathcal{U}(r)$ from $g(r) = -\dfrac{d\mathcal{U}}{dr}$.

Self-Gravitating Lattice Gas at Equilibrium

Thermal and hydrostatic equilibrium:

- EOS: $\dfrac{p(r)V_\mathrm{c}}{k_\mathrm{B}T} = -\ln\Big(1-\rho(r)\Big)$,
- EOM: $\dfrac{dp}{dr} = \dfrac{m_\mathrm{c}}{V_\mathrm{c}}\rho(r)g(r), \quad g(r) = -\dfrac{Gm_\mathrm{in}}{r^{\mathcal{D}-1}}, \quad m_\mathrm{in} = \dfrac{m_\mathrm{c}}{V_\mathrm{c}}\displaystyle\int_0^r dr'\left(\mathcal{A}_\mathcal{D}r'^{\mathcal{D}-1}\right)\rho(r')$.

Scaled variables:

- $\hat{r} \doteq \dfrac{r}{r_\mathrm{s}}, \quad \hat{p} \doteq \dfrac{p}{p_\mathrm{s}}, \quad \hat{T} \doteq \dfrac{k_\mathrm{B}T}{p_\mathrm{s}V_\mathrm{c}}, \quad \hat{\mathcal{U}} \doteq \dfrac{\mathcal{U}}{p_\mathrm{s}V_\mathrm{c}}$,
- $r_\mathrm{s}^\mathcal{D} = \dfrac{NV_\mathrm{c}\mathcal{D}}{\mathcal{A}_\mathcal{D}}, \quad p_\mathrm{s} = \dfrac{\mathcal{A}_\mathcal{D}G}{2\mathcal{D}}\dfrac{m_\mathrm{c}^2}{V_\mathrm{c}^2}r_\mathrm{s}^2$.

Relations between profiles:

- EOS & EOM: $\hat{p}(\hat{r}) = -\hat{T}\ln\big(1-\rho(\hat{r})\big), \qquad \dfrac{d\hat{p}}{d\hat{r}} = -2\mathcal{D}\rho(\hat{r})\displaystyle\int_0^{\hat{r}} d\hat{r}'\left(\frac{\hat{r}'}{\hat{r}}\right)^{\mathcal{D}-1}\rho(\hat{r}')$,
- $\hat{U}(\hat{r}) = \hat{T}\ln\left(\dfrac{1-\rho(\hat{r})}{1-\rho(0)}\,\dfrac{\rho(0)}{\rho(\hat{r})}\right)$. Reference value: $\hat{\mathcal{U}}(0) = 0$.

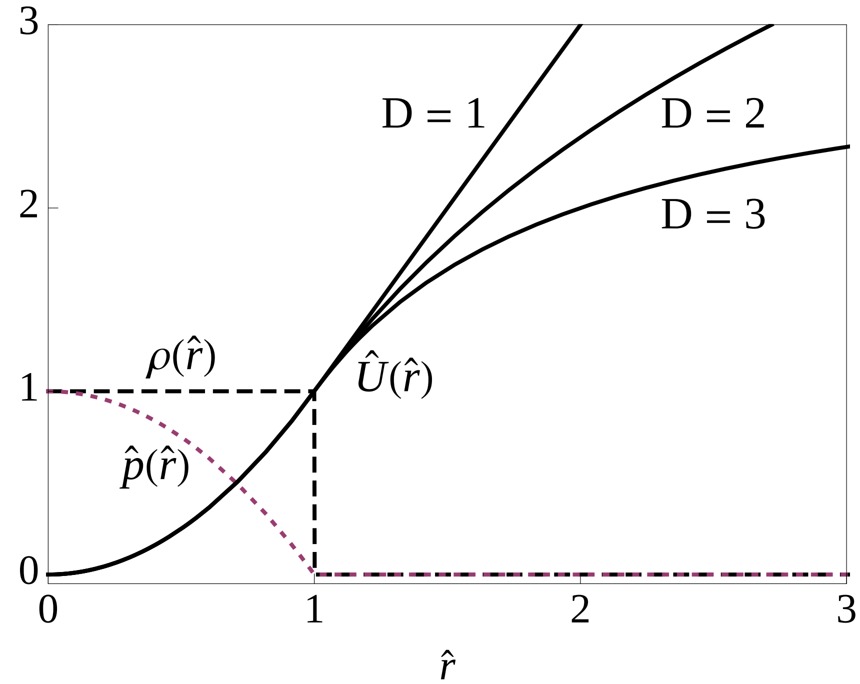
Profiles at $T = 0$: Solid Cluster

- $\rho(\hat{r}) = \begin{cases} 1 & : 0 \leq \hat{r} < 1, \\ 0 & : \hat{r} > 1, \end{cases}$

- $\hat{p}(\hat{r}) = \begin{cases} 1 - \hat{r}^2 & : 0 \leq \hat{r} < 1, \\ 0 & : \hat{r} > 1, \end{cases}$

- $\hat{\mathcal{U}}(\hat{r}) = \hat{r}^2 \quad : 0 \leq \hat{r} \leq 1,$

$$\hat{\mathcal{U}}(\hat{r}) = \left\{ \begin{array}{ll} 2\hat{r} - 1 & : \mathcal{D} = 1, \\ 2\ln\hat{r} + 1 & : \mathcal{D} = 2, \\ 3 - 2/\hat{r} & : \mathcal{D} = 3, \end{array} \right\} \quad \hat{r} \geq 1.$$



Escape velocity finite in $\mathcal{D} > 2$.

Stable clusters of finite mass exist

- at all $\hat{T}$ $(\mathcal{D} = 1)$,
- at $\hat{T} < \hat{T}_c$ $(\mathcal{D} = 2)$,
- at $\hat{T} = 0$ only $(\mathcal{D} = 3)$.

Differential Equations for Profiles

Analysis of EOS & EOM at $\hat{T} > 0$:

- density: $\rho'' + \frac{\mathcal{D}-1}{\hat{r}}\rho' - \frac{1-2\rho}{\rho(1-\rho)}(\rho')^2 + \frac{2\mathcal{D}}{\hat{T}}\rho^2(1-\rho) = 0, \quad \rho'(0) = 0,$
- pressure: $\hat{p}'' + \frac{\mathcal{D}-1}{\hat{r}}\hat{p}' - \frac{1-\rho}{\hat{T}\rho}(\hat{p}')^2 + 2\mathcal{D}(\rho)^2 = 0, \quad \hat{p}'(0) = 0,$
- potential: $\hat{\mathcal{U}}'' + \frac{\mathcal{D}-1}{\hat{r}}\hat{\mathcal{U}}' - 2\mathcal{D}\rho = 0, \quad \hat{\mathcal{U}}(0) = \hat{\mathcal{U}}'(0) = 0,$

  with $\rho(\hat{r}) = 1 - e^{-\hat{p}(\hat{r})/\hat{T}}; \quad \rho(\hat{r}) = \frac{1}{e^{(\hat{\mathcal{U}}(\hat{r})-\hat{\mu})/\hat{T}} + 1}, \quad \hat{\mu} \doteq \frac{\mu}{p_\mathrm{s}V_\mathrm{c}} = -\hat{T}\ln\left(\frac{1-\rho(0)}{\rho(0)}\right).$

Additional boundary conditions:

- density: $\mathcal{D}\int_0^\infty d\hat{r}\,\hat{r}^{\mathcal{D}-1}\rho(\hat{r}) = 1,$
- pressure: $2\mathcal{D}(\mathcal{D}-1)\int_0^\infty d\hat{r}\,\hat{r}^{2\mathcal{D}-3}\hat{p}(\hat{r}) = 1,$
- in $\mathcal{D} = 1$ only: $\hat{p}(0) = 1, \quad \rho(0) = 1 - e^{-1/\hat{T}}.$

Ideal Classical Gas Limit (1)

ICG assumption: low density $\rho(\hat{r}) \ll 1$ realized throughout the cluster.

ODE for density profile: $\dfrac{\rho''}{\rho} + \dfrac{\mathcal{D}-1}{\hat{r}}\dfrac{\rho'}{\rho} - \left(\dfrac{\rho'}{\rho}\right)^2 + \dfrac{2\mathcal{D}}{\hat{T}}\rho = 0,$

with boundary conditions as in ILG.

Analytic solutions for finite clusters:

- $\mathcal{D} = 1$: $\rho(\hat{r})_{\mathrm{ICG}} = \dfrac{1}{\hat{T}}\,\mathrm{sech}^2\left(\dfrac{\hat{r}}{\hat{T}}\right),$

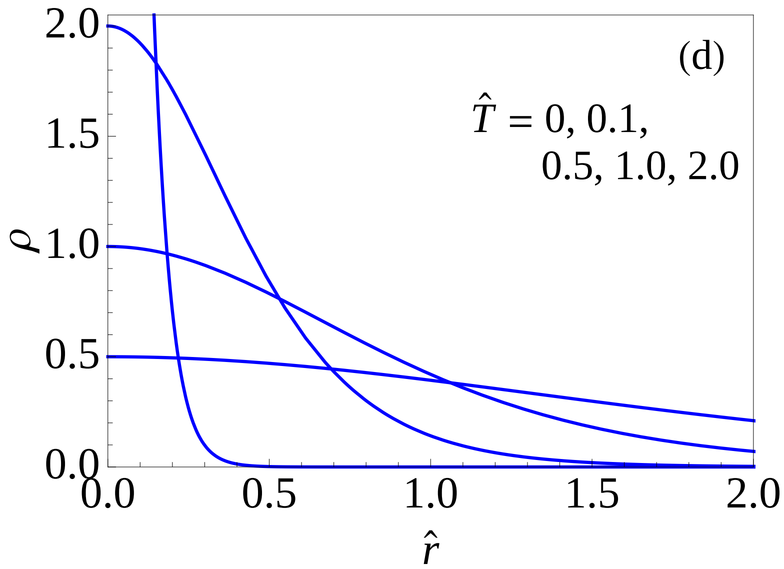
  solution normalized at all $\hat{T}$,

  exponential decay: $\rho(\hat{r}) \sim e^{-2\hat{r}/\hat{T}}$.

- $\mathcal{D} = 2$: $\rho(\hat{r})_{\mathrm{ICG}} = \dfrac{4c}{\hat{T}}\left[1 + 2c\left(\dfrac{\hat{r}}{\hat{T}}\right)^2\right]^{-2},$

  solution normalizable only for $\hat{T}_{\mathrm{c}} = \dfrac{1}{2}$,

  power-law decay: $\rho(\hat{r}) \sim \hat{r}^{-4}$.



ICG in $\mathcal{D} = 3$ confined to sphere of radius $\hat{R} \doteq R/r_{\rm s}$.

Rescaled variables: $\bar{r} \doteq \hat{r}/\hat{R}, \quad \bar{\rho} \doteq \hat{R}^3\rho, \quad \bar{T} \doteq \hat{R}\hat{T}.$

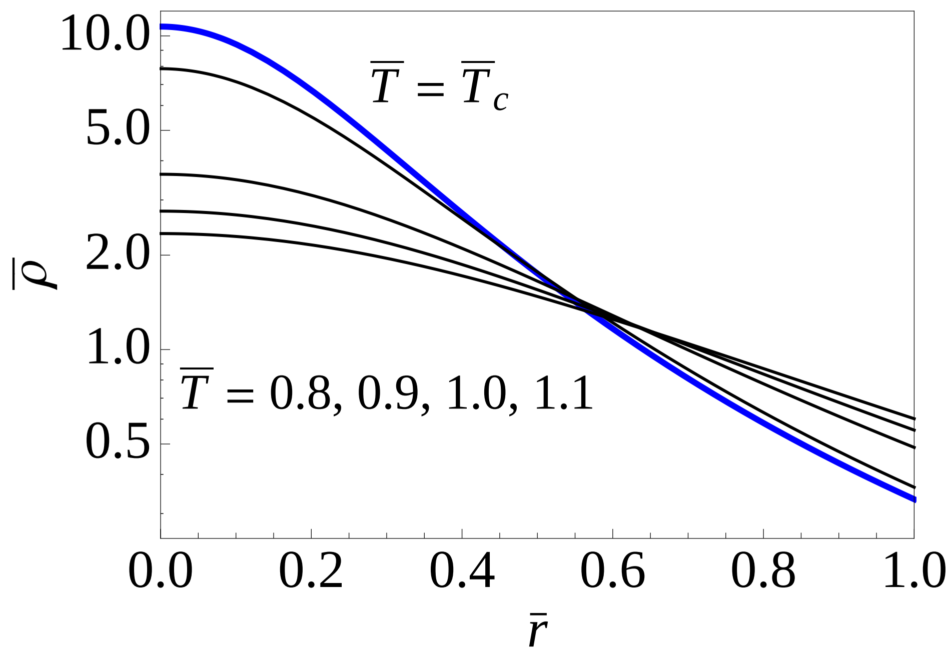
ODE: $\frac{\bar{\rho}''}{\bar{\rho}} + \frac{2}{\bar{r}}\frac{\bar{\rho}'}{\bar{\rho}} - \left(\frac{\bar{\rho}'}{\bar{\rho}}\right)^2 + \frac{6\bar{\rho}}{\bar{T}} = 0, \quad \bar{\rho}'(0) = 0, \quad 3\int_0^1 d\bar{r}\,\bar{r}^2\bar{\rho}(\bar{r}) = 1.$

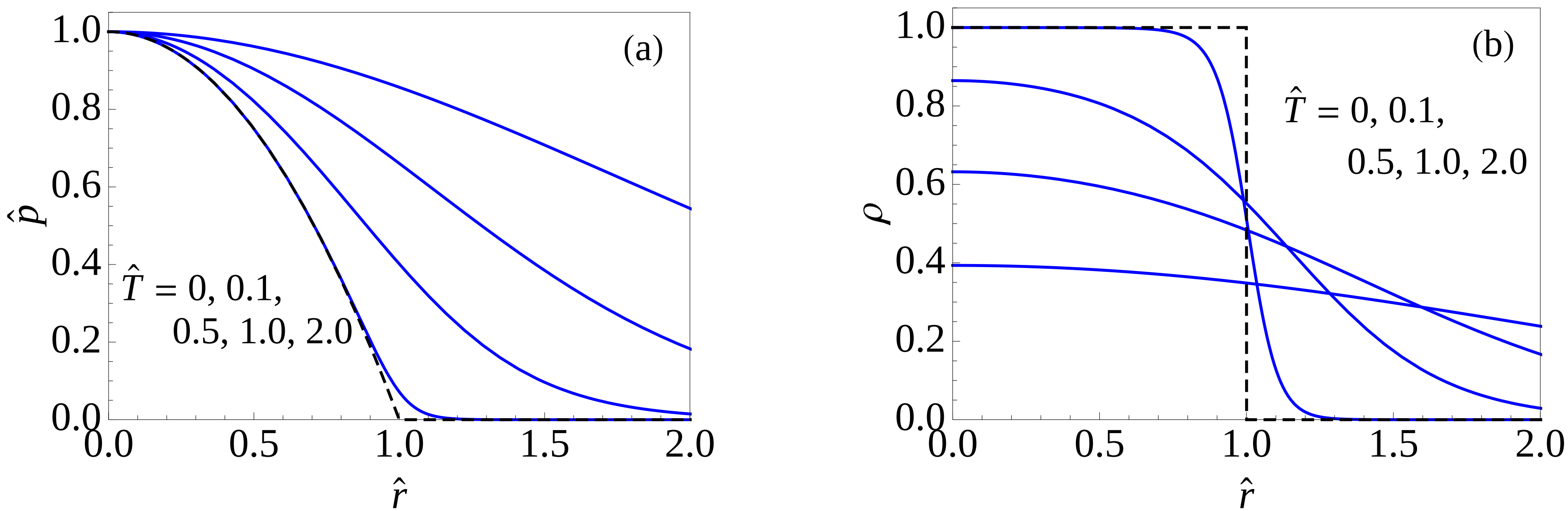
Normalizable solutions exist for $\bar{T} \geq \bar{T}_{\rm C} = 0.794422\ldots$

Vega and Sanchez (2006) predict

- spinodal point: $\bar{T}_{\rm C} = 0.79442$,
- transition point: $\bar{T}_{\rm T} = 0.8215$,

based on computer simulations.



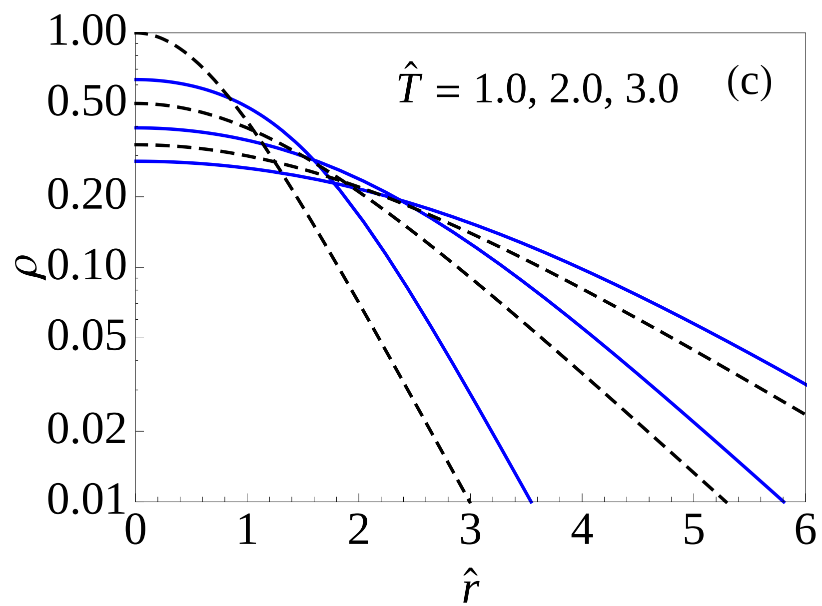


Analytic ICG profile:

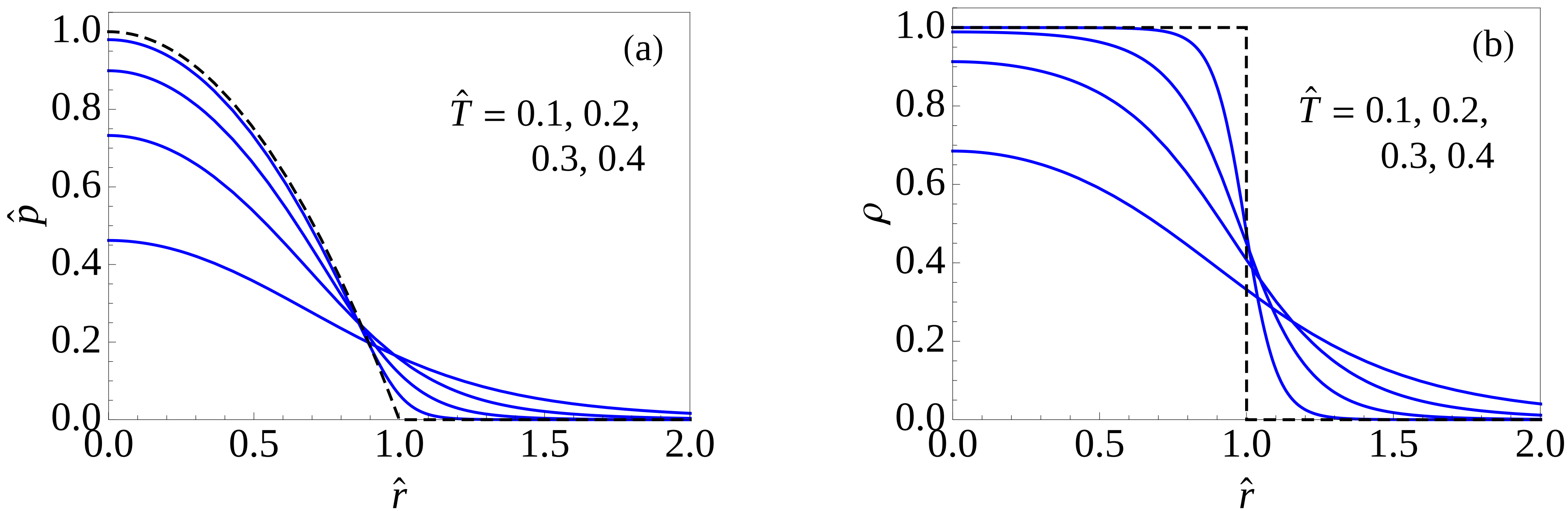
$$\rho(\hat{r})_{\mathrm{ICG}} = \frac{1}{\hat{T}} \,\mathrm{sech}^2\left(\frac{\hat{r}}{\hat{T}}\right)$$

Exact ILG asymptotics:

$$\rho(\hat{r})_{\mathrm{as}} \sim e^{-2\hat{r}/\hat{T}} \quad : \; \hat{r} \gg 1$$



ILG Profiles for $\mathcal{D} = 2$: Unconfined Space

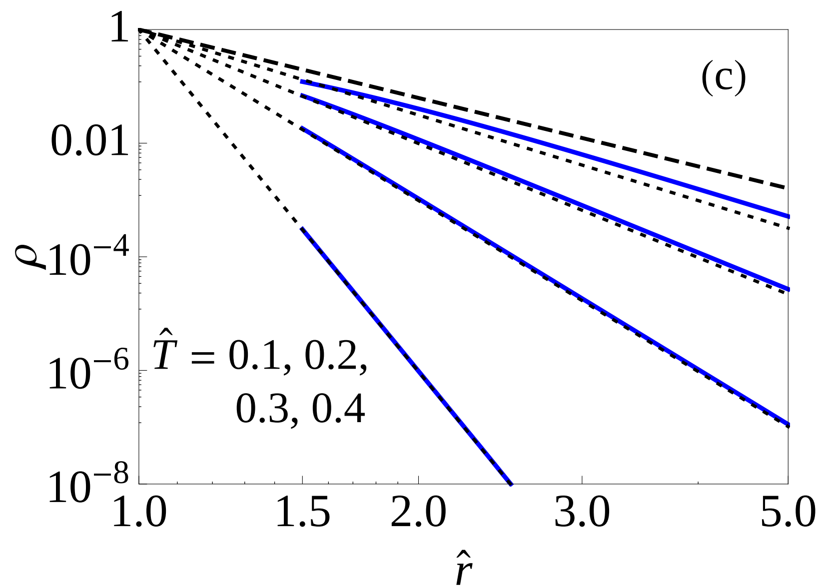


Critical ILG profile: $\hat{T} \lesssim \hat{T}_c = \frac{1}{2}$

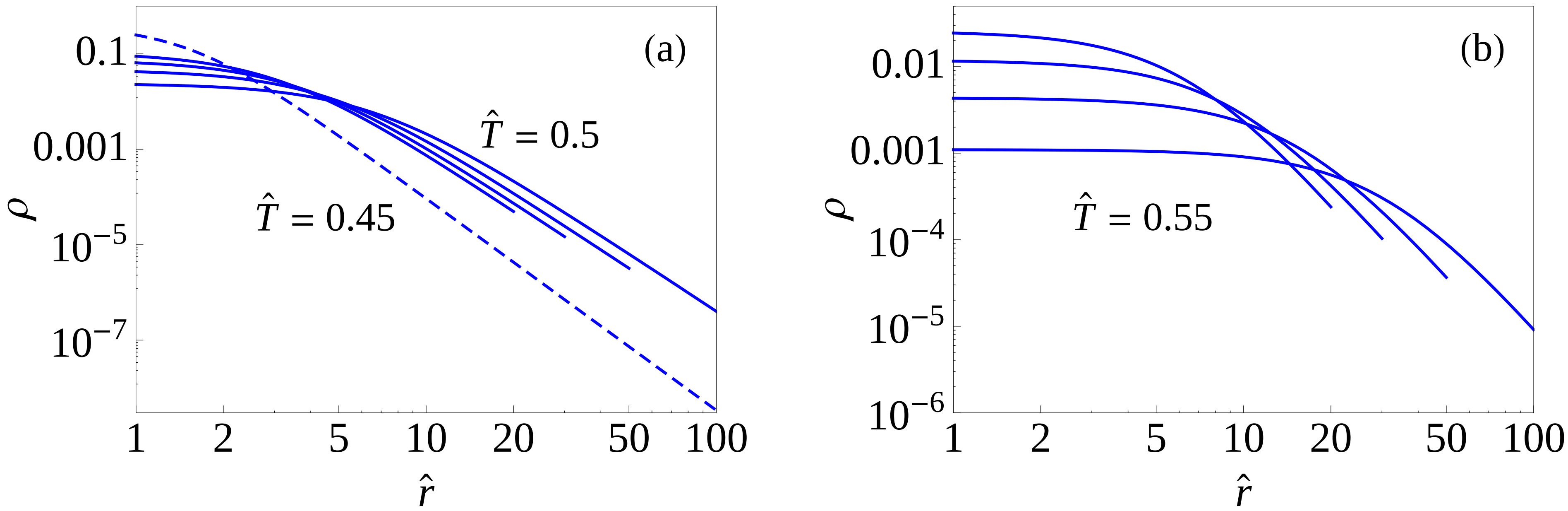
$$\rho(\hat{r})_{\mathrm{ICG}} = \frac{4c}{\hat{T}} \left[1 + 2c \left(\frac{\hat{r}}{\hat{T}} \right)^2 \right]^{-2}$$

Exact ILG asymptotics: $0 \leq \hat{T} \leq \hat{T}_c$

$$\rho(\hat{r})_{\mathrm{as}} \sim \hat{r}^{-2/\hat{T}} \quad : \ \hat{r} \gg 1$$



ILG Profiles for $\mathcal{D} = 2$: Confined Space

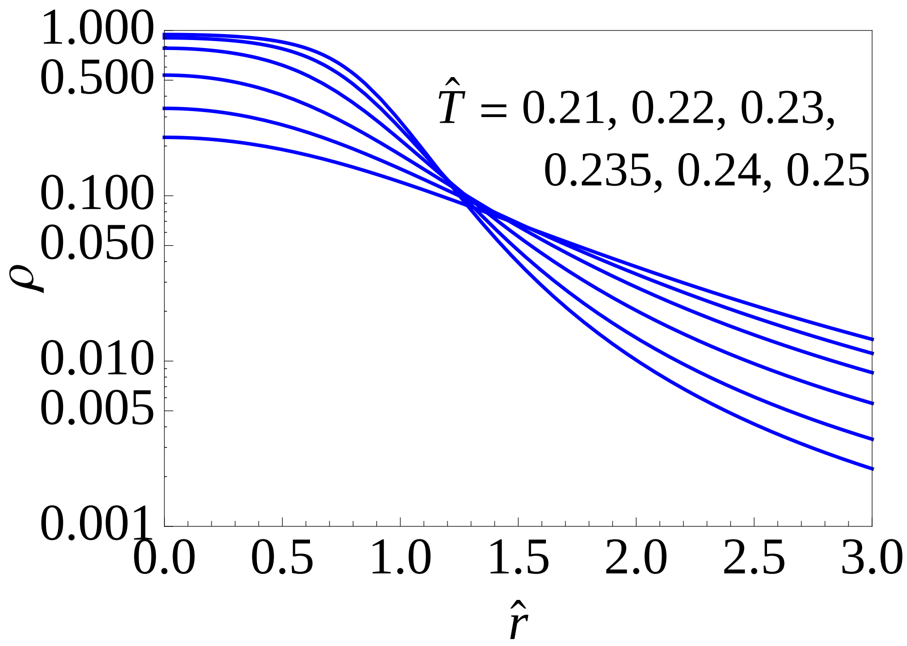


ILG in $\mathcal{D} = 2$ confined to disk of radius $\hat{R} = 20, 30, 50, 100$.

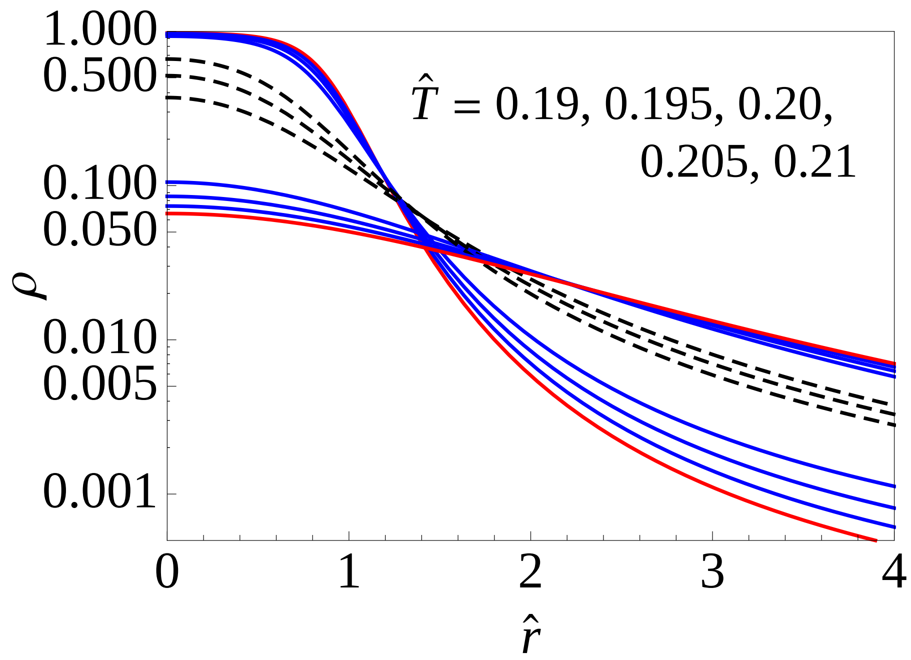
- $\hat{T} = 0.45$: stable cluster,
- $\hat{T} = 0.5$: stability limit,
- $\hat{T} = 0.55$: unstable cluster.

ILG Profiles for $\mathcal{D} = 3$: Confined Space

ILG in $\mathcal{D} = 3$ confined to sphere of radius $\hat{R}$.



$\hat{R} = 3$

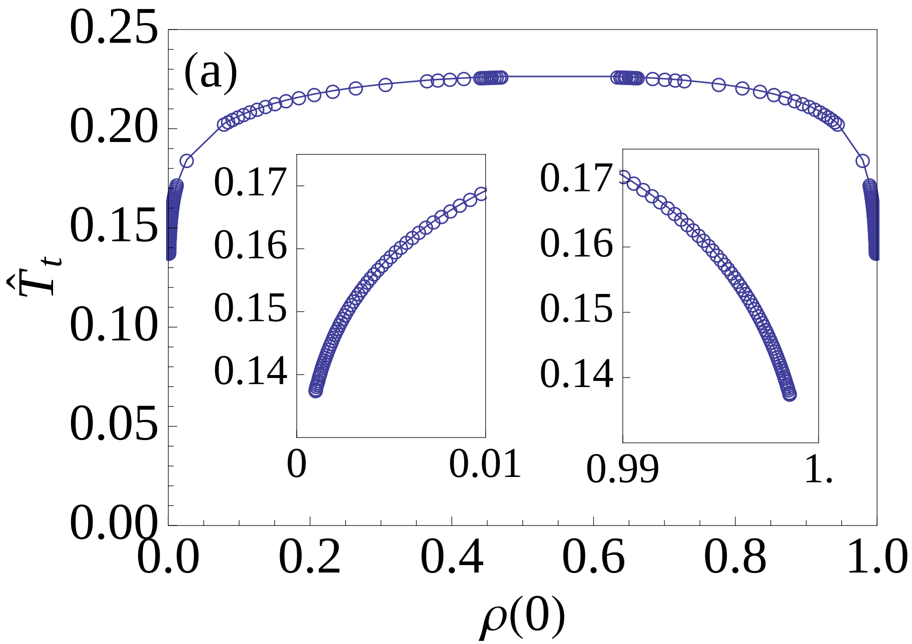


$\hat{R} = 4$

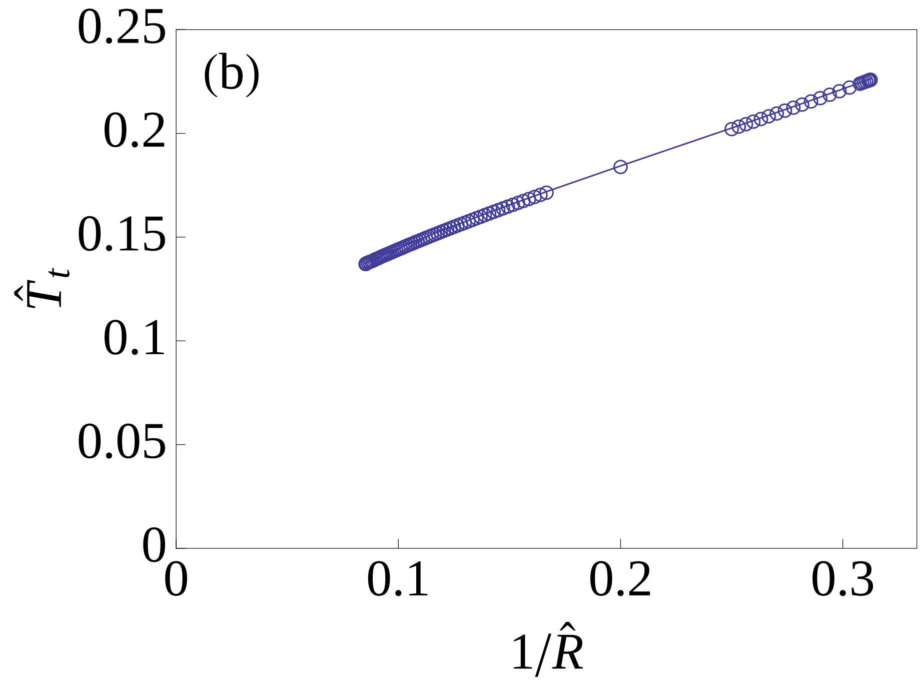
- one normalizable solution at all $\hat{T}$,
- gradual cluster formation,
- no phase transition.

- three normalizable solutions in $\hat{T}$ interval,
- two mechanically stable solutions,
- free energy minimum switches,
- first-order phase transition.

ILG Profiles for $\mathcal{D} = 3$: First-Order Transition



Guggenheim plot



transition temperature $\hat{T}_{\rm t}$

- critical temperature: $\hat{T}_{\rm c} \lesssim 0.23$,
- critical radius: $\hat{R}_{\rm c} \gtrsim 3.1$

Open ILG Clusters

- Closed system: finite mass, confined or unconfined space.

$$\rho'' + \frac{\mathcal{D}-1}{\hat{r}}\rho' - \frac{1-2\rho}{\rho(1-\rho)}(\rho')^2 + \frac{2\mathcal{D}}{\hat{T}}\rho^2(1-\rho) = 0, \quad \rho'(0) = 0, \quad \mathcal{D}\int_0^{\hat{R}} d\hat{r}\,\hat{r}^{\mathcal{D}-1}\rho(\hat{r}) = 1.$$

$$\hat{r} \doteq \frac{r}{r_\mathrm{s}}, \quad \hat{p} \doteq \frac{p}{p_\mathrm{s}}, \quad \hat{T} \doteq \frac{k_\mathrm{B}T}{p_\mathrm{s}V_\mathrm{c}}, \quad \hat{\mathcal{U}} \doteq \frac{\mathcal{U}m_\mathrm{c}}{p_\mathrm{s}V_\mathrm{c}}, \quad r_\mathrm{s}^{\mathcal{D}} = \frac{NV_\mathrm{c}\mathcal{D}}{\mathcal{A}_{\mathcal{D}}}, \quad p_\mathrm{s} = \frac{\mathcal{A}_{\mathcal{D}}G}{2\mathcal{D}}\frac{m_\mathrm{c}^2}{V_\mathrm{c}^2}r_\mathrm{s}^2.$$

- Open system: finite or infinite mass, unconfined space.

$$\tilde{\rho}'' + \frac{\mathcal{D}-1}{\tilde{r}}\tilde{\rho}' - \frac{1-2\rho_0\tilde{\rho}}{\tilde{\rho}(1-\rho_0\tilde{\rho})}\left(\tilde{\rho}'\right)^2 + \tilde{\rho}^2(1-\rho_0\tilde{\rho}) = 0, \quad \tilde{\rho}(0) = 1, \quad \tilde{\rho}'(0) = 0.$$

$$\tilde{r} \doteq \sqrt{\frac{2\mathcal{D}\rho_0}{\hat{T}}}\,\hat{r}, \quad \tilde{\rho} \doteq \frac{\rho}{\rho_0}.$$

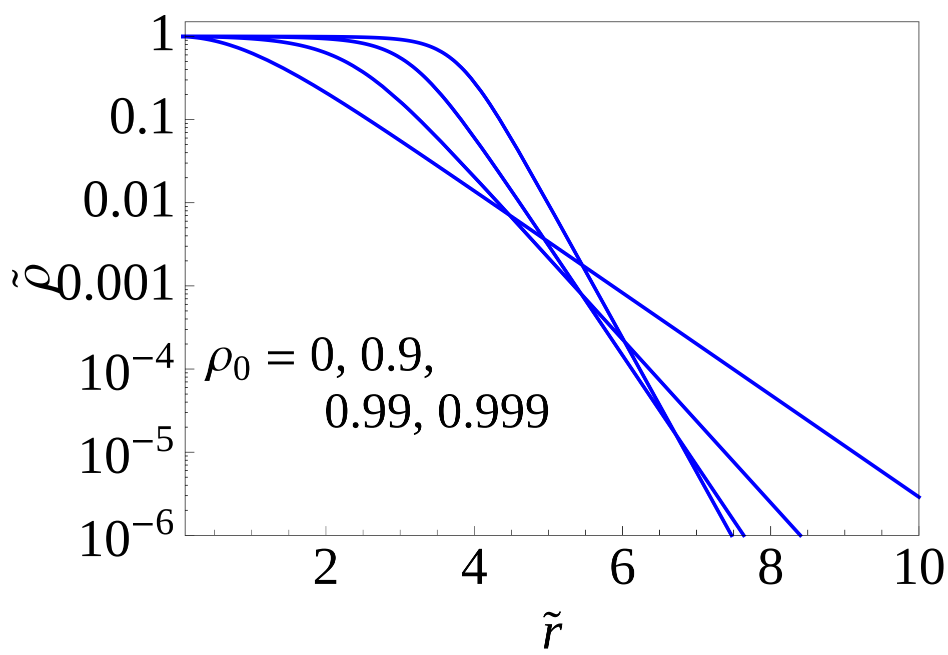
- ICG limit $\rho_0 \to 0$:

$$\frac{\tilde{\rho}''}{\tilde{\rho}} + \frac{\mathcal{D}-1}{\tilde{r}}\frac{\tilde{\rho}'}{\tilde{\rho}} - \left(\frac{\tilde{\rho}'}{\tilde{\rho}}\right)^2 + \tilde{\rho} = 0 \quad \Rightarrow \tilde{\rho}(\tilde{r}) \sim \begin{cases} e^{-\sqrt{2}\,\tilde{r}} & : \ \mathcal{D} = 1, \\ \tilde{r}^{-4} & : \ \mathcal{D} = 2, \\ \tilde{r}^{-2} & : \ \mathcal{D} = 3. \end{cases}$$

Open ILG Cluster in $\mathcal{D} = 1$

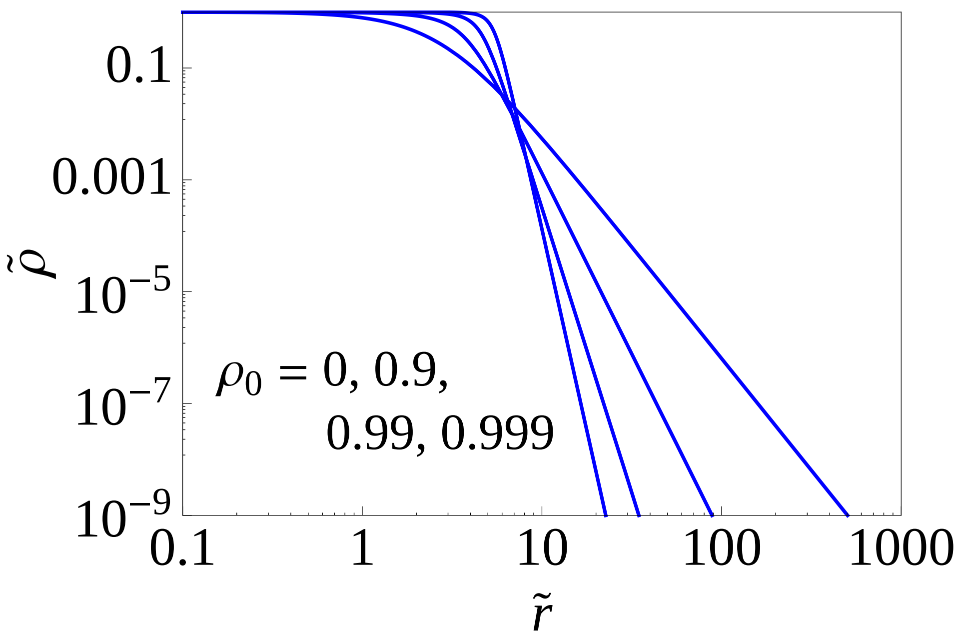
Use inverse function $\tilde{r}(\tilde{\rho})$ for analytic solution:

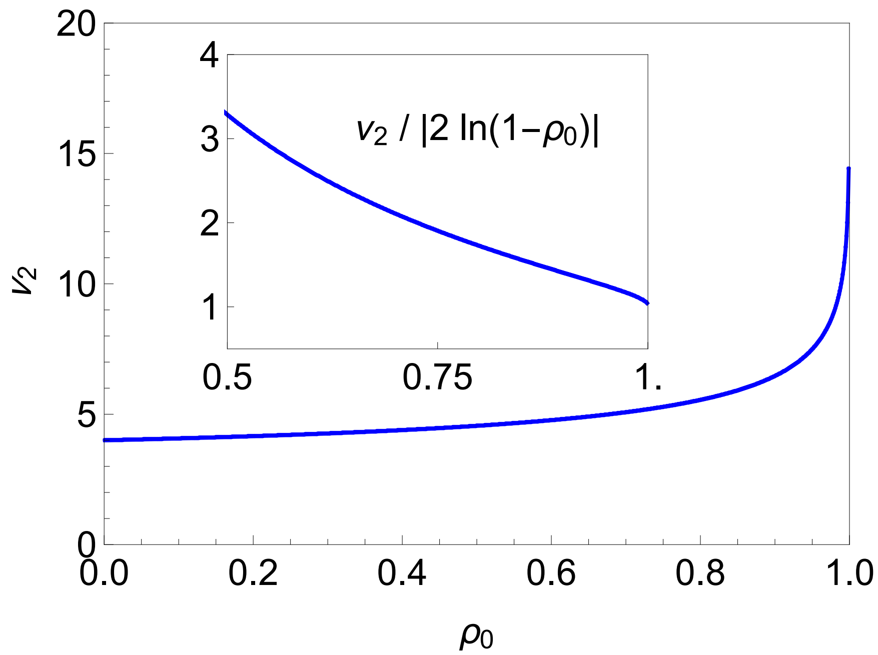
- $\tilde{r}'' + \dfrac{1-2\rho}{\rho(1-\rho)}\tilde{r}' - \dfrac{1}{\rho_0}\rho^2(1-\rho)(\tilde{r}')^3 = 0, \quad \rho = \rho_0\tilde{\rho}, \quad \tilde{r}(\rho_0) = 0, \quad \tilde{r}'(\rho_0) = -\infty$
- $\tilde{r}(\tilde{\rho}) = \sqrt{\dfrac{\rho_0}{2}} \displaystyle\int_{\tilde{\rho}}^{1} \dfrac{d\tilde{\rho}'}{\tilde{\rho}'(1-\rho_0\tilde{\rho}')\sqrt{\ln\frac{1-\rho_0\tilde{\rho}'}{1-\rho_0}}} \quad : 0 \le \tilde{\rho} \le 1$
- $\tilde{\rho}(\tilde{r})_{\mathrm{as}} \sim e^{-\nu_1(\rho_0)\tilde{r}}$
- $\nu_1(\rho_0) = \sqrt{-\dfrac{2}{\rho_0}\ln(1-\rho_0)}$
- $\tilde{\rho}(\tilde{r})_{\mathrm{ICG}} = \mathrm{sech}^2\left(\dfrac{\tilde{r}}{\sqrt{2}}\right)$
- finite mass, exponential decay law



Numerical solution:

- $\tilde{\rho}(\tilde{r})_{\rm as} \sim \tilde{r}^{-\nu_2(\rho_0)}, \quad \nu_2(\rho_0) \stackrel{\rho_0 \to 1}{\longrightarrow} -2\ln(1-\rho_0)$
- $\tilde{\rho}(\tilde{r})_{\rm ICG} = \dfrac{1}{\left(1+\tilde{r}^2/8\right)^2}$
- finite mass, algebraic decay law

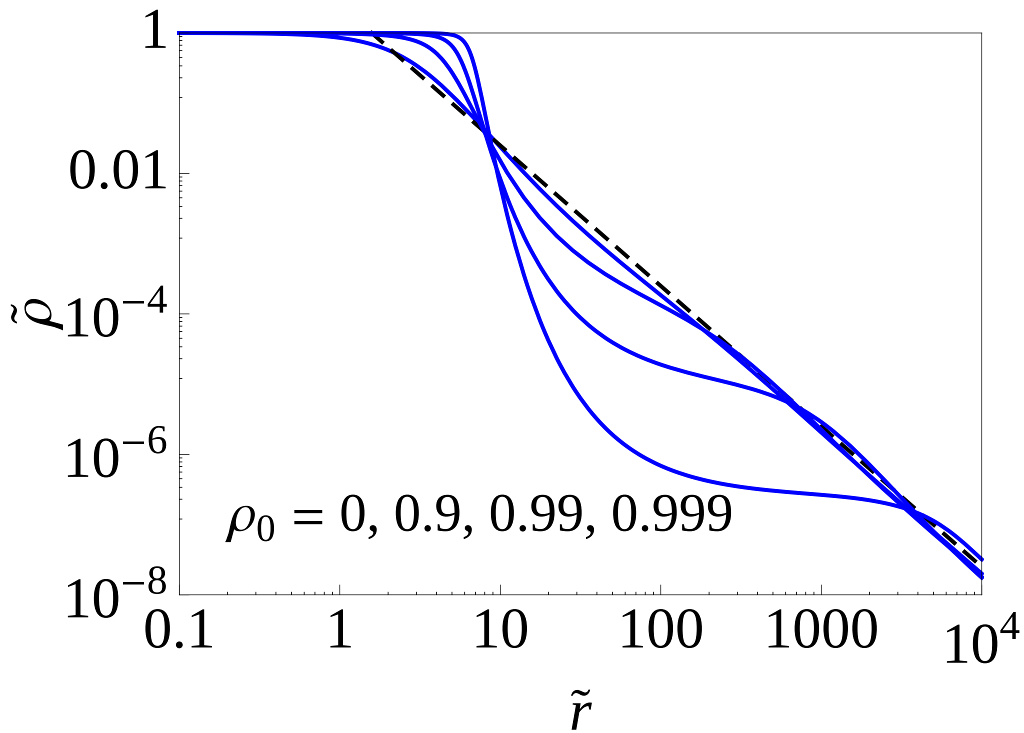




Open ILG Cluster in $\mathcal{D} = 3$

Numerical solution:

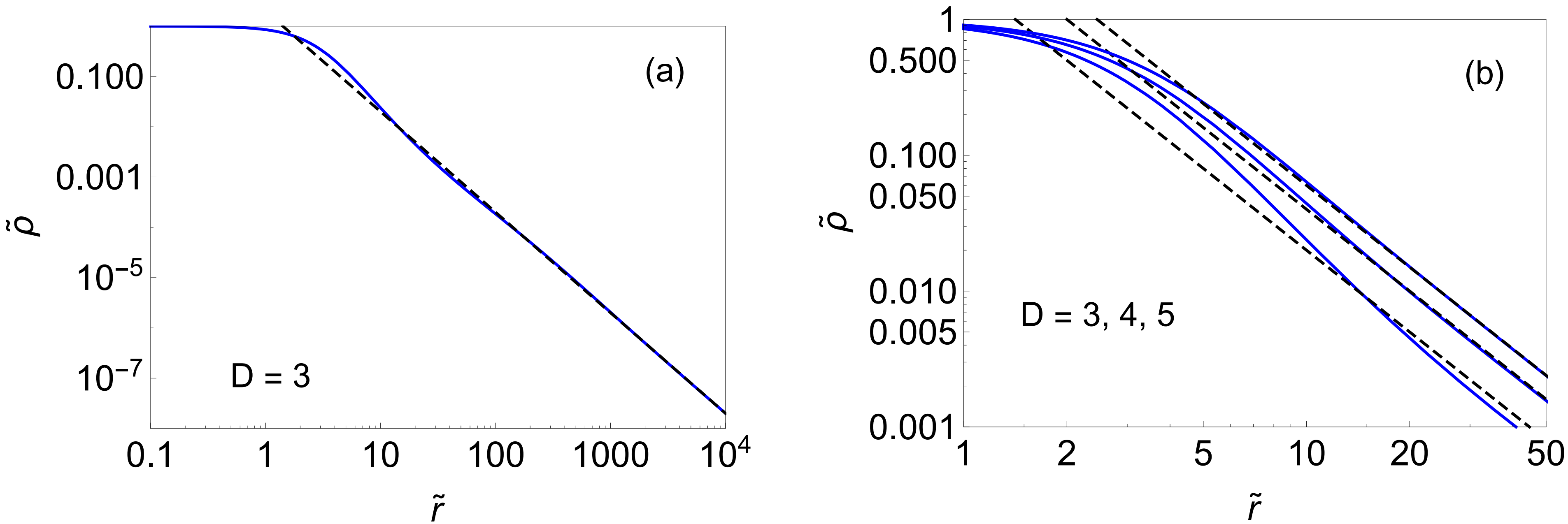
- $\tilde{\rho}(\tilde{r})_{\rm as} \sim \dfrac{2}{\tilde{r}^2}$
- infinite mass, algebraic decay law
- ICG: Bonnor-Ebert sphere
  - gaseous core
  - gaseous halo
- ILG: emerging horizon
  - condensed matter core
  - gaseous shell
  - gaseous halo



ICG Asymptotics in $\mathcal{D} \geq 3$

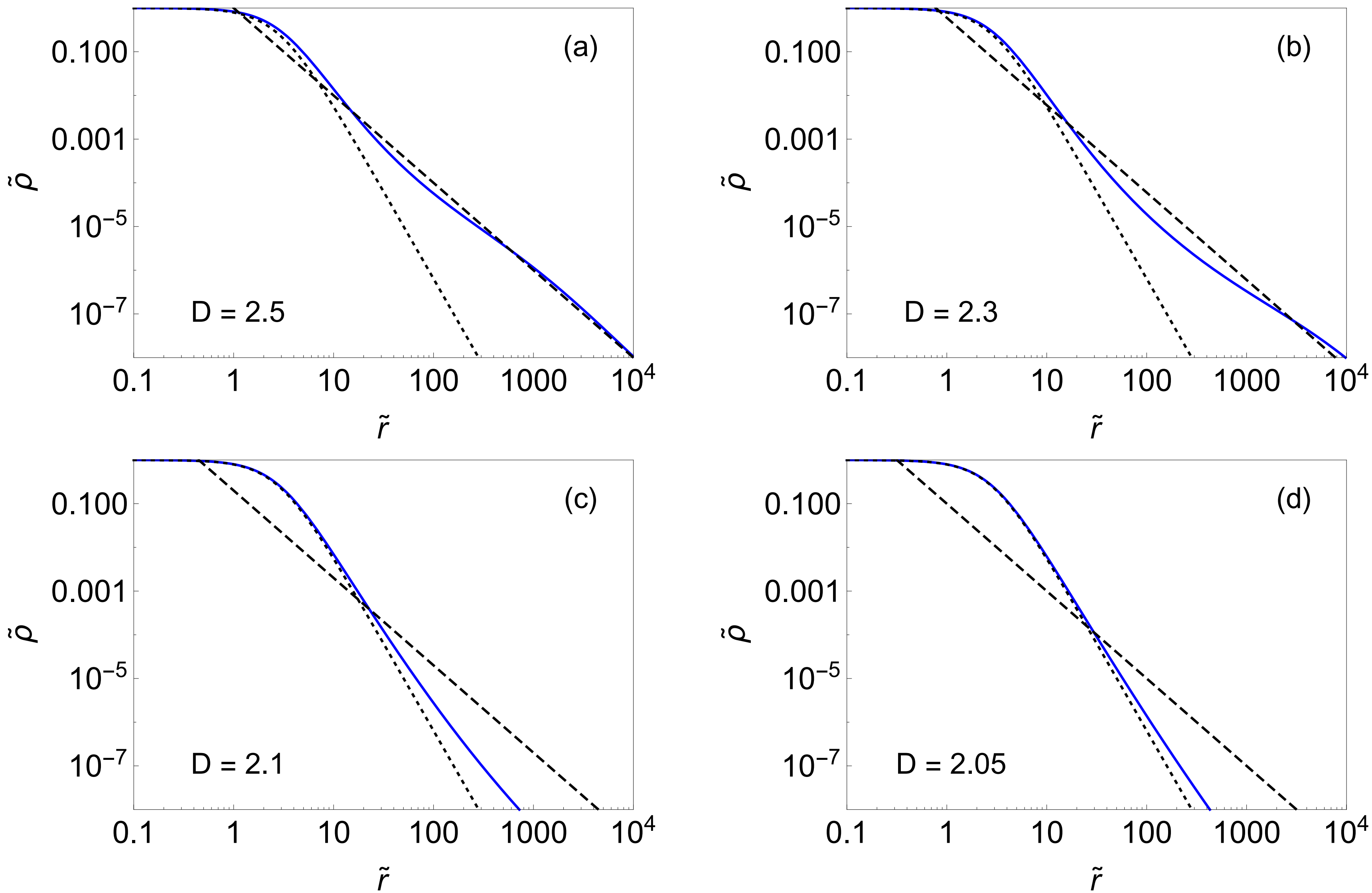
$$\frac{\tilde\rho''}{\tilde\rho} + \frac{\mathcal{D}-1}{\tilde r}\frac{\tilde\rho'}{\tilde\rho} - \left(\frac{\tilde\rho'}{\tilde\rho}\right)^2 + \tilde\rho = 0 \quad \Rightarrow \tilde\rho(\tilde r) \sim \begin{cases} e^{-\sqrt{2}\,\tilde r} & : \mathcal{D} = 1, \\ \tilde r^{-4} & : \mathcal{D} = 2, \\ \tilde r^{-2} & : \mathcal{D} = 3. \end{cases}$$

- $\tilde\rho(\tilde r) \sim a\tilde r^{-\alpha}; \qquad \alpha = 2, \quad a = 2(\mathcal{D}-2).$
- infinite mass, algebraic decay law



ICG Asymptotics in $2 \leq \mathcal{D} \leq 3$

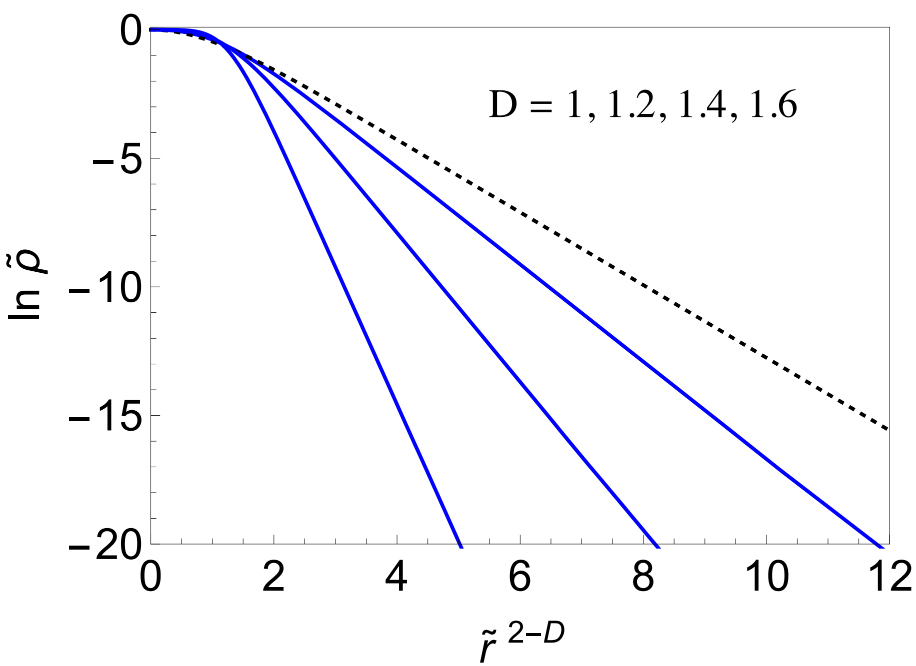
- infinite mass, crossover between algebraic decay laws

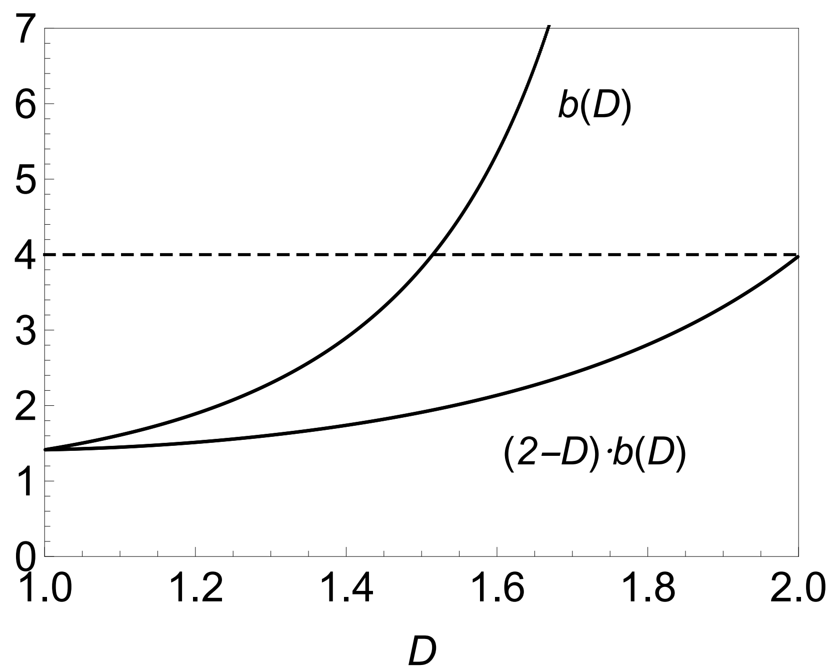


ICG Asymptotics in $1 \leq \mathcal{D} \leq 2$

$$\frac{\tilde{\rho}''}{\tilde{\rho}} + \frac{\mathcal{D}-1}{\tilde{r}}\frac{\tilde{\rho}'}{\tilde{\rho}} - \left(\frac{\tilde{\rho}'}{\tilde{\rho}}\right)^2 + \tilde{\rho} = 0 \quad \Rightarrow \quad \tilde{\rho}(\tilde{r}) \sim \begin{cases} e^{-\sqrt{2}\,\tilde{r}} & : \mathcal{D} = 1, \\ \tilde{r}^{-4} & : \mathcal{D} = 2, \\ \tilde{r}^{-2} & : \mathcal{D} = 3. \end{cases}$$

- $\tilde{\rho}(\tilde{r}) \sim \exp\left(-b\tilde{r}^{\beta}\right)$; $\quad \beta = 2 - \mathcal{D}, \quad \lim_{\mathcal{D}\to 1} b(\mathcal{D}) = \sqrt{2}, \quad \lim_{\mathcal{D}\to 2} \beta(\mathcal{D})b(\mathcal{D}) = 4.$
- finite mass, stretched exponential decay law





Outlook

Rotating clusters:

- Competition between gravitational and centrifugal forces.
- Multivalued functional relation between $\vec{L}$ and $\vec{\omega}$.
- instabilities of density profiles:
  - binary configurations in $\mathcal{D} = 1$,
  - ring configurations in $\mathcal{D} = 2$,
  - effects of hysteresis.

Quantum gas clusters:

- ODE for fugacity profile $z(r)$.
- Comparison between FD and BE gases, CS generalization.
- FD Pauli principle vs ILG hardcore repulsion.
- Relativistic effects.
- BE condensation vs gravitational collapse.
- FD cluster drifting into region of magnetic field.