Structure of shock waves

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The structure of a plane stationary shock wave in an arbitrary liquid or gas at large distances from the front is studied. All the hydrodynamic quantities approach their equilibrium values in proportion to $z^{-3/2}$ with increasing distance z. Such a slow decrease is due to the contribution of long-wave hydrodynamic fluctuations.

The problem of the structure of shock waves is solved in general form for an arbitrary fluid only in the case of waves of sufficiently low intensity,[1] when the thickness of the wave front is large and the hydrodynamic approach can be used. In this case, all the quantities (density, velocity of the fluid, etc.) approach their equilibrium values exponentially with increasing the distance from the front, and the thickness of the front decreases with increasing intensity of the wave, until it is equal to the characteristic length parameters in the system. In gases, such a parameter is the free path length of the particles, and in liquids it is the interatomic distance. The structure of strong shock waves is therefore strongly dependent on the details of the microscopic structure of the system and cannot naturally be explained in general form.

Of significant interest in the given case is the problem of the wave structure at large distances from the front. This problem has been treated repeatedly in the case of gases[2,3] and in an arbitrary system in the presence of slowly relaxing parameters (for example, slow chemical reactions, rotational relaxation etc; see, for example, the book of Zel'dovich and Raĭzer[4]). It is significant that the limiting law according to which all the quantities approach their equilibrium values at large distances has always been found to be exponential. It is therefore clear that if there exists a mechanism that leads to a power law, then this is the principal mechanism at sufficiently large distances.

In the present work we shall show that account of hydrodynamic fluctuations leads to a very slow power law, which has a universal form for shock waves of any intensity, in any media. We have noted previously[5,6] that the hydrodynamic fluctuations are a fundamental source of nonlocality in hydrodynamics and represent a mechanism of propagation of excitations in a liquid at distances considerably exceeding the usual dissipation penetration depth. In the case of shock waves, the phenomenon can be represented in the following way. Thermal fluctuations with large wavelengths are always present in a liquid; these include, for example, acoustic fluctuations. In a homogeneous liquid, these fluctuations are in thermodynamic equilibrium. In the presence of a shock wave, only those fluctuations which are incident on the wave are in equilibrium. Reflected and refracted fluctuations, as a consequence of the Doppler effect and the anisotropic character of the shock wave, are nonequilibrium near the front. This nonequilibrium character is dispersed at distances of the order of the free path length of the fluctuations.

From the fact that fluctuations of the path length can be large at sufficiently low frequencies (for example, the absorption of sound oscillations is proportional to the square of the frequency), the dispersal of the nonequilibrium fluctuations takes place according to a very slow power law. This law is easily determined from the following simple considerations. Inasmuch as the free path length is inversely proportional to the square of the frequency, the maximum frequency of the excitations that penetrate to a distance z from the front of the shock wave is inversely proportional to the square root of z. The effect produced by such excitations is proportional to the corresponding volume of phase space:

$$k^3 \sim \omega^3 \sim z^{-3/2},$$

where k is the wave vector and ω the frequency. It is therefore clear that all the hydrodynamic quantities (velocity of the liquid, density, pressure, etc) will approach their equilibrium values like $z^{-3/2}$ at large distances. We shall calculate below the coefficients in this law and show that they are all expressed in terms of thermodynamic quantities and the kinetic coefficients of the medium.

1. The solution of the problem of the structure of a plane stationary shock wave is conveniently carried out in a set of coordinates in which the wave front is at rest. Furthermore, let the z axis be directed along the normal to the front, such that the direction of the velocity of the liquid coincides with the positive z direction. Inasmuch as the liquid advances on the shock wave with supersonic speed in the region in front of the shock (in the half-space $z < 0$), all the excitations here propagate in the direction of the front. Therefore the fluctuations are at equilibrium in the region ahead of the wave front. We are interested in the region $z > 0$ behind the wave front, where the fluctuations are nonequilibrium, inasmuch as the liquid here moves away from the front with subsonic speed and therefore excitations exist that are both incident on the shock wave and departing from it.

If we neglect the fluctuations, then the shock wave should be considered as a discontinuity plane ($z = 0$) and all the hydrodynamic quantities in the regions $z > 0$ (the density ρ, the velocity of the liquid **v**, etc) and $z < 0$ (the density ρ', the velocity $\mathbf{v}'$, etc) are certain constants that do not depend on the coordinates and the time. The presence of fluctuations causes the appearance of small corrections ($\delta\rho$, $\delta\mathbf{v}$ and so forth), to all the quantities, but then corrections are oscillatory functions of the time and therefore vanish upon averaging.

The effect of interest to us arises in the second approximation in the amplitude of the fluctuations. In this approximation, the pressure p, the entropy per unit mass of liquid σ, and the velocity can be represented in the form of sums:

$$p+\delta p+p^{(2)},\quad \sigma+\delta\sigma+\sigma^{(2)},\quad \mathrm{v}+\delta\mathrm{v}+\mathrm{v}^{(2)},$$

where the first terms are constant in the zeroth approximation, the second are the oscillating first-order corrections mentioned above, and the third are the second order quantities that we are interested in.

Carrying out an expansion with accuracy to quantities of second order, we can easily determine the fluctuation corrections to all the other hydrodynamic quantities. For example, the oscillating correction to the density is equal to $\delta\rho = (\delta p/c^2) + r\delta\sigma$, where c is the velocity of sound, $r = (\partial\rho/\partial\sigma)_p$, while the correction of second order can be written down in the form

$$\rho^{(2)}=\frac{1}{c^2}p^{(2)}+r\sigma^{(2)}-\frac{1}{c^3}\left(\frac{\partial c}{\partial p}\right)_\sigma(\delta p)^2+\frac{1}{2}\left(\frac{\partial r}{\partial \sigma}\right)_p(\delta\sigma)^2-\frac{2}{c^3}\left(\frac{\partial c}{\partial \sigma}\right)_p\delta p\,\delta\sigma. \tag{1}$$

To calculate the time averages

$$\overline{\sigma^{(2)}},\ \overline{p^{(2)}},\ \overline{v^{(2)}}=\overline{v_z^{(2)}}$$

it is more convenient to start from the equations of hydrodynamics, written in the form of the conservation laws for the mass, energy and the z component of the momentum. If we take the time average of the continuity equation then, inasmuch as the average value of the time derivative of the density vanishes, we obtain the condition for constancy of the average value of the z component of the mass flux

$$\overline{j_z^{(2)}}(z)=\text{const}.$$

This average value can be expressed in analogy with (1), in terms of $\overline{p^{(2)}}$, $\overline{v^{(2)}}$, $\overline{\sigma^{(2)}}$:

$$\overline{j_z^{(2)}}=\rho\overline{v^{(2)}}+v\overline{\rho^{(2)}}+\overline{\delta\rho\delta v_z}=\rho\overline{v^{(2)}}+\frac{v}{c^2}\overline{p^{(2)}}+rv\overline{\sigma^{(2)}}+\frac{1}{c^2}\overline{\delta p\delta v_z}+r\overline{\delta\sigma\delta v_z}$$
$$-\frac{v}{c^3}\left(\frac{\partial c}{\partial p}\right)_\sigma\overline{(\delta p)^2}+\frac{v}{2}\left(\frac{\partial r}{\partial \sigma}\right)_p\overline{(\delta\sigma)^2}-\frac{2v}{c^3}\left(\frac{\partial c}{\partial \sigma}\right)_p\overline{\delta p\delta\sigma}. \tag{2}$$

In similar fashion, we obtain formulas for the z component of the energy flux $Q_z = \rho v_z(w + v^2/2)$ (w is the heat function per unit mass) and momentum $\Pi_{zz} = p + \rho v_z^2$, the average values of which are constants by virtue of the equations of hydrodynamics:

$$\overline{Q_z^{(2)}}=v\overline{p^{(2)}}+\rho vT\overline{\sigma^{(2)}}+\rho v^2\overline{v^{(2)}}+(w+{}^1/_2v^2)\,\overline{j_z^{(2)}}+\frac{vT}{c^2}\overline{\delta p\delta\sigma}$$
$$+vT\left(r+\frac{\rho}{2c_p}\right)\overline{(\delta\sigma)^2}+(1+M^2)\,\overline{\delta p\delta v_z}+(rv^2+\rho T)\,\overline{\delta\sigma\delta v_z}$$
$$+\frac{3}{2}\rho v\,\overline{(\delta v_z)^2}+\frac{M}{2\rho c}\overline{(\delta p)^2}+\frac{\rho v}{2}\{\overline{(\delta v_x)^2}+\overline{(\delta v_y)^2}\}, \tag{3}$$

$$\overline{\Pi_{zz}^{(2)}}=(1+M^2)\overline{p^{(2)}}+rv^2\overline{\sigma^{(2)}}+2\rho v\overline{v^{(2)}}-\frac{M^2}{c}\left(\frac{\partial c}{\partial p}\right)_\sigma\overline{(\delta p)^2}$$
$$+\frac{v^2}{2}\left(\frac{\partial r}{\partial \sigma}\right)_p\overline{(\delta\sigma)^2}-\frac{2M^2}{c}\left(\frac{\partial c}{\partial \sigma}\right)_p\overline{\delta p\,\delta\sigma}+\rho\overline{(\delta v_z)^2}+\frac{2M}{c}\overline{\delta p\,\delta v_z}+2rv\overline{\delta\sigma\,\delta v_z}, \tag{4}$$

where T is the temperature of the liquid, $M = v/c$ is the Mach number, and c_p the specific heat at constant pressure.

If we now solve the set of three equations (2)—(4) relative to the three unknown quantities $\overline{p^{(2)}}$, $\overline{\sigma^{(2)}}$, $\overline{v^{(2)}}$, then these quantities will be expressed linearly (with constant coefficients) in terms of the fluxes

$$\overline{j_z^{(2)}},\ \overline{Q_z^{(2)}},\ \overline{\Pi_{zz}^{(2)}}$$

and the mean-square fluctuations $\overline{(\delta p)^2}$, $\overline{\delta p\delta v_z}$ and so on. Since the fluxes do not depend on the z coordinate, it is clear that the formulas obtained here will determine the coordinate dependence of the pressure, entropy and velocity if the coordinate dependence of the mean-square fluctuations is known.

We then obtain the following for the difference between the real and asymptotic (as $z \to \infty$) values of the hydrodynamic quantities

$$\bar p(z)-p(\infty)\equiv\overline{p^{(2)}}(z)-\overline{p^{(2)}}(\infty)=(1-M^2)^{-1}\Big\{-M\Big[\frac{1}{c}\left(\frac{\partial c}{\partial p}\right)_\sigma$$
$$+\frac{r}{2\rho T}\Big]\overline{(\delta p)^2}+v^2\left[\frac{1}{2}\left(\frac{\partial r}{\partial \sigma}\right)_p-r\left(\frac{r}{\rho}+\frac{1}{2c_p}\right)\right]\overline{(\delta\sigma)^2}-\left(\rho+\frac{rv^2}{2T}\right)\overline{(\delta v_z)^2}$$
$$-\frac{rv^2}{2T}[\overline{(\delta v_x)^2}+\overline{(\delta v_y)^2}]-\frac{rv}{\rho T}\overline{\delta p\,\delta v_z}-rv\overline{\delta\sigma\,\delta v_z}$$
$$-M^2\left[\frac{2}{c}\left(\frac{\partial c}{\partial \sigma}\right)_p+\frac{r}{\rho}\right]\overline{\delta p\,\delta\sigma}\Big\}, \tag{5}$$

$$\bar\sigma(z)-\sigma(\infty)=-\frac{\overline{(\delta p)^2}}{2T\rho^2c^2}-\left(\frac{r}{\rho}+\frac{1}{2c_p}\right)\overline{(\delta\sigma)^2}-\frac{\overline{(\delta \mathbf{v})^2}}{2T}$$
$$-\frac{\overline{\delta p\,\delta\sigma}}{\rho c^2}-\frac{\overline{\delta p\,\delta v_z}}{\rho vT}-\frac{\overline{\delta\sigma\,\delta v_z}}{v}, \tag{6}$$

$$\bar v(z)-v(\infty)=-\frac{1}{\rho v}[\bar p(z)-p(\infty)]-\frac{1}{v}\overline{(\delta v_z)^2}-\frac{1}{\rho c^2}\overline{\delta p\,\delta v_z}-\frac{r}{\rho}\overline{\delta\sigma\,\delta v_z}, \tag{7}$$

where we should substitute for the mean-square fluctuations their nonequilibrium parts, i.e., the differences between their real and asymptotic (equilibrium) values. Thus the problem is reduced to the calculation of the coordinate dependence of the mean-square fluctuations.

2. Any small excitation in a liquid is a superposition of acoustic, entropy, and vortex waves. The departure of the pressure δp from the equilibrium value is due to the presence of sound waves only, and therefore it can be written down in the form

$$\delta p=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\{p_\mathbf{k}\exp[i\mathbf{kr}-i\omega_s(\mathbf{k})t]+p_\mathbf{k}^*\exp[-i\mathbf{kr}+i\omega_s(\mathbf{k})t]\}, \tag{8}$$

where $p_\mathbf{k}$ are certain amplitudes, $\omega = \mathbf{v}\cdot\mathbf{k} + c|\mathbf{k}|$ is the frequency of the sound wave with wave vector $\mathbf{k}$ in a liquid moving with velocity $\mathbf{v}$. If the liquid is in the state of thermodynamic equilibrium, then the amplitudes $p_\mathbf{k}$ satisfy the condition

$$\overline{p_\mathbf{k}p_{\mathbf{k}'}^*}=(p^2)_\mathbf{k}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{9}$$

where $(p^2)_\mathbf{k} = \rho Tc^2/2$. Here

$$\overline{(\delta p)^2}=\int\frac{d^3\mathbf{k}}{(2\pi)^3}\rho Tc^2,$$

as it ought to be according to the thermodynamic theory of fluctuations.

If the fluctuations are not at equilibrium because of the inhomogeneity of the liquid, but the characteristic scale of the inhomogeneity is large in comparison with the wavelength of the fluctuations (in our case, the characteristic dimension is of the order of the free path length of the fluctuations, which exceeds the wavelength appreciably), then Eq. (9) holds as before; however, the quantities $(p^2)_\mathbf{k}$ are not equal to the equilibrium value cited above, and are slowly changing functions of the coordinates. These quantities play the role of the distribution function of the acoustic fluctuations.

The deviation $\delta\sigma$ of the entropy from its equilibrium value can be written in similar fashion:

$$\delta\sigma=\int_{k_z>0}\frac{d^3\mathbf{k}}{(2\pi)^3}\{\sigma_\mathbf{k}\exp[i\mathbf{kr}-i\mathbf{v}\mathbf{k}t]+\sigma_\mathbf{k}^*\exp[-i\mathbf{kr}+i\mathbf{v}\mathbf{k}t]\}, \tag{10}$$

$$\overline{\sigma_\mathbf{k}\sigma_{\mathbf{k}'}^*}=(\sigma^2)_\mathbf{k}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{11}$$

where $(\sigma^2)_k = c_p/\rho$ at equilibrium. In contrast with Eqs. (8) the integration in (10) is carried out only over positive k_z, because the frequency $\omega = \mathbf{v}\cdot\mathbf{k}$ for the entropy waves is uniquely determined by the wave vector, while for sound waves there are two possible values of the frequency for a given **k**.

The vortex waves have two polarizations for a given k, corresponding to two possible directions of the velocity vector in the plane perpendicular to **k**. The deviation of the velocity $\delta\mathbf{v}$ from a constant value, which is due to the vortex waves, is equal to

$$\delta\mathbf{v} = \int_{k_z>0} \frac{d^3\mathbf{k}}{(2\pi)^3} \{\mathbf{n}_\alpha v_{\mathbf{k}\alpha} \exp[i\mathbf{k}\mathbf{r} - i\mathbf{v}\mathbf{k}t] + \mathbf{n}_\alpha v_{\mathbf{k}\alpha}^* \exp[-i\mathbf{k}\mathbf{r} + i\mathbf{v}\mathbf{k}t]\}, \quad (12)$$

where the index α runs over the values 1 and 2, $\mathbf{n}_1$ and $\mathbf{n}_2$ are mutually perpendicular vectors in the plane perpendicular to **k**, the amplitudes v_k satisfy the condition

$$\overline{v_{\mathbf{k}\alpha} v_{\mathbf{k}'\beta}^*} = (v_\alpha v_\beta)_\mathbf{k} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \quad (13)$$

while in the equilibrium state

$$(v_\alpha v_\beta)_\mathbf{k} = (T/\rho)\delta_{\alpha\beta}.$$

In the following, it will be convenient to assume that the vector $\mathbf{n}_1$ lies in a plane perpendicular to the vector **k** and the z axis, while the vector $\mathbf{n}_2$ is perpendicular to this plane.

In the relations (8), (10), (12) for the fluctuations of the hydrodynamic quantities, it is necessary to separate the part associated with the nonequilibrium waves that move away from the shock wave, since the incident waves are equilibrium ones and do not make a contribution to the nonequilibrium part of the mean-square fluctuations of interest to us.

The outgoing waves at $z > 0$ are all the entropy and vortex waves and also the sound waves with $\partial\omega/\partial k_z = c(M + \cos\theta_s) > 0$, where $\cos\theta_s = k_z/|\mathbf{k}|$. The amplitudes of the outgoing waves are uniquely determined by the amplitudes of the incident waves if the corresponding coefficients of reflection, refraction, and transmission are known. The calculation of the coefficients can be carried out with the help of the known boundary conditions on the discontinuity surface at $z = 0$. The researches of D'yakov[7] and Kontorovich[8] were devoted to the last question, and it is easy to obtain on their basis formulas for all the coefficients of interest. These formulas are given in Appendix 1 of this paper.

The deviation of the entropy from the equilibrium value, which is due to the presence of entropy waves outgoing from the shock wave, can be represented in the following form:

$$\delta\sigma = \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{W_\sigma^{(A)} A_\mathbf{k} \exp[i\mathbf{k}_\sigma\mathbf{r} - i\omega_A(\mathbf{k})t - \varkappa_\sigma z] + \text{c.c.}\}, \quad (14)$$

where summation is carried out over all types of incident waves, A_k are the amplitudes of incident waves (i.e., p_k for sound waves, σ_k for entropy waves, v_{k1} for vortex waves), $W_\sigma^{(A)}$ are the transformation coefficients of incident waves into entropy waves. Integration in Eq. (14) is carried out over the wave vector **k** of the incident waves, $\mathbf{k}_s = \mathbf{k}_\sigma(\mathbf{k})$ is the wave vector of the departing entropy wave, i.e., the wave which has the same value of frequency and projection of the wave vector on the plane $z = 0$ as the incident wave. We have also taken into account the presence of damping of the waves, which is described by the last term in the exponential in (14) where $\varkappa_\sigma = (\kappa/\rho v c_p)k_\sigma^2$, κ is the thermal conductivity of the liquid.

In similar fashion, one can write down the deviation of the pressure from its equilibrium value, which is due to sound waves outgoing from the discontinuity:

$$\delta p = \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{W_s^{(A)} A_\mathbf{k} \exp[i\mathbf{k}_s\mathbf{r} - i\omega_A(\mathbf{k})t - \varkappa_s z] - \text{c.c.}\}, \quad (15)$$

where $W_s^{(A)}$ are the transformation coefficients of incident waves into sound, $\mathbf{k}_s = \mathbf{k}_s(\mathbf{k})$ is the wave vector of the outgoing sound wave, $\varkappa_s = (\gamma k_s^2/2\rho c)(M + k_{sz}/k_s)$, $\gamma = \frac{4}{3}\eta + \zeta + \kappa(1/c_V - 1/c_p)$, η, ζ is the first and second viscosity coefficients, and c_V is the specific heat of the liquid at constant volume.

The velocity fluctuations are caused by the vortical and sound waves. We therefore have for them

$$\delta v_z = \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{\frac{\cos\theta_s}{\rho c} W_s^{(A)} A_\mathbf{k} \exp[i\mathbf{k}_s\mathbf{r} - i\omega_A(\mathbf{k})t - \varkappa_s z] + \text{c.c.}\right\}$$
$$+ \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{-\sin\theta W_v^{(A)} A_\mathbf{k} \exp[i\mathbf{k}_v\mathbf{r} - i\omega_A(\mathbf{k})t - \varkappa_v z] + \text{c.c.}\},$$
$$\delta v_\parallel = \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{\frac{\sin\theta_s}{\rho c} W_s^{(A)} A_\mathbf{k} \exp[i\mathbf{k}_s\mathbf{r} - i\omega_A(\mathbf{k})t - \varkappa_s z] + \text{c.c.}\right\} \quad (16)$$
$$+ \sum_A \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{\cos\theta W_v^{(A)} A_\mathbf{k} \exp[i\mathbf{k}_v\mathbf{r} - i\omega_A(\mathbf{k})t - \varkappa_v z] + \text{c.c.}\}.$$

Here

$$\delta v_\parallel = [(\delta v_x)^2 + (\delta v_y)^2]^{1/2}$$

is the projection of the deviation of the velocity from a constant value on the plane $z = 0$, $\mathbf{k}_v = \mathbf{k}_v(\mathbf{k})$ is the wave vector of the vortical wave, $\varkappa_v = (\eta/\rho v)k_v^2$, $W_v^{(A)}$ the coefficients of transformation of the incident waves into vortical waves, $\cos\theta = k_{vz}/k_v = \omega/vk_v$, $\cos\theta_{sz}/k_s$. We have taken it into account that the oscillations of velocity and pressure in the sound waves are connected with one another by the relation $\delta\mathbf{v} = \mathbf{k}_s\delta p/\rho c k_s$.

It should be emphasized that in Eqs. (14)–(16), among the possible incident waves, one should consider vortical waves polarized only in the plane of incidence. Only such waves are transformed at the discontinuity into other types of waves. The vortical waves which are polarized perpendicular to the plane of incidence, as is clear from considerations of symmetry, pass through the discontinuity without change in amplitude. The change in velocity in the region $z > 0$ due to them is equal to

$$\delta\mathbf{v} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \{v_{\mathbf{k}2}\mathbf{n}_2 \exp[i\mathbf{k}_v\mathbf{r} - ivk_{vz}t - \varkappa_v z] + \text{c.c.}\}, \quad (17)$$

where v_{k2} is the amplitude of the vortical wave with polarization 2 incident on the discontinuity from the region $z < 0$.

3. Using Eqs. (14)–(17), it is easy to calculate all the mean-square fluctuations of interest. For example, let us consider the quantity $\overline{(\delta\sigma)^2}$. Squaring Eqs. (14) and averaging by means of the relation

$$\overline{A_\mathbf{k} A_{\mathbf{k}'}^*} = (A^2)_\mathbf{k} (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$$

(the different types of incident waves are statistically independent of one another), we obtain, after transition from integration over **k** to integration over $\mathbf{k}_\sigma$,

$$\overline{(\delta\sigma)^2} = \int_{k_{\sigma z}>0} \frac{d^3\mathbf{k}_\sigma}{(2\pi)^3} \sum_A \frac{\partial k_z}{\partial k_{\sigma z}} 2|W_\sigma^{(A)}|^2 (A^2)_\mathbf{k} \exp(-2\varkappa_\sigma z). \quad (18)$$

In this equation, we have still not taken the equilibrium fluctuations into account in proper fashion. Actually, as $z \to \infty$, the quantities $(\sigma^2)_k$ determined by the expression (18) tend to zero exponentially. In reality, however, as $z \to \infty$, the difference in the quantities $(\sigma^2)_k$ and their equilibrium values c_p/ρ tend to zero. To obtain the correct result for the nonequilibrium part of $\overline{(\delta\sigma)^2}$, we should subtract the quantity $\overline{2c_p/\rho}$ from the expression before the exponential in (18) (a detailed justification of this procedure is given in Appendix 2):

$$\overline{(\delta\sigma)^2} = \int\limits_{k_{\sigma z}>0} \frac{d^3\mathbf{k}_\sigma}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k_{\sigma z}} 2|W_\sigma^{(A)}|^2 (A^2)_\mathbf{k} - \frac{2c_p}{\rho}\right\}\exp(-2\varkappa_\sigma z). \quad (19)$$

We find all the remaining mean-square fluctuations in similar fashion:

$$\overline{(\delta p)^2} = \int \frac{d^3\mathbf{k}_s}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k_{sz}} 2|W_s^{(A)}|^2 (A^2)_\mathbf{k} - \rho T c^2\right\}\exp(-2\varkappa_s z),$$

$$\overline{(\delta v_z)^2} = \int \frac{d^3\mathbf{k}_s}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k_{sz}} 2|W_s^{(A)}|^2 (A^2)_\mathbf{k} \frac{\cos^2\theta_s}{\rho^2 c^2} - \frac{\cos^2\theta_s}{\rho} T\right\}\exp(-2\varkappa_s z)$$
$$+ \int \frac{d^3\mathbf{k}_v}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k_{vz}} 2|W_v^{(A)}|^2 (A^2)_\mathbf{k} \sin^2\theta - \sin^2\theta \frac{2T}{\rho}\right\}\exp(-2\varkappa_v z),$$

$$\overline{(\delta v_x)^2} + \overline{(\delta v_y)^2} = \int \frac{d^3\mathbf{k}_s}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k_{sz}} 2|W_s^{(A)}|^2 (A^2)_\mathbf{k} \frac{\sin^2\theta_s}{\rho^2 c^2} - \sin^2\theta_s \frac{T}{\rho}\right\}$$
$$\times \exp(-2\varkappa_s z) + \int \frac{d^3\mathbf{k}_v}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k_{vz}} 2|W_v^{(A)}|^2 (A^2)_\mathbf{k} \cos^2\theta\right.$$
$$\left. - \cos^2\theta \frac{2T}{\rho}\right\}\exp(-2\varkappa_v z),$$

$$\overline{\delta p \delta v_z} = \int \frac{d^3\mathbf{k}_s}{(2\pi)^3}\left\{\sum_A \frac{\partial k_z}{\partial k} 2|W_s^{(A)}|^2 (A^2)_\mathbf{k} \frac{\cos\theta_s}{\rho c} - 2Tc\cos\theta_s\right\}\exp(-2\varkappa_s z),$$

$$\overline{\delta\sigma\delta v_z} = -\int \frac{d^3\mathbf{k}_v}{(2\pi)^3}\sum_A \frac{\partial k_z}{\partial k_{vz}} \sin\theta\, 2(A^2)_\mathbf{k} \operatorname{Re}(W_\sigma^{(A)} W_v^{(A)*})\exp[-(\varkappa_\sigma + \varkappa_v)z]. \quad (20)$$

The mean value $\overline{\delta p\delta\sigma}$ is equal to zero in this approximation. This is due to the fact that the pressure and entropy fluctuations are connected as a whole with the sound and entropy waves, respectively. The considered mean value is proportional to the integral over $\mathbf{k}$ of the product of the amplitudes of the sound and entropy waves, which arise from the same incident wave. As a result, an integral of the form (20) is obtained for $\delta p\delta\sigma$, with only this difference that the integral contains the factor $\exp[iz(k_{sz} - k_{\sigma z})]$ which, by virtue of the fact that the sound and entropy waves are characterized by different dependences $\omega(\mathbf{k})$, is a rapidly oscillating function of $\mathbf{k}$ for large z and therefore the corresponding integral is very small.

In connection with the fact that there is also a similar factor $\exp[iz(k_{\sigma z} - k_{vz})]$ in the expression for $\overline{\delta\sigma\delta v}$, which, however, is equal to unity in the given case, inasmuch as the $\omega(\mathbf{k})$ dependence is the same for entropy and vortical waves, and therefore $\mathbf{k}_\sigma = \mathbf{k}_v$.

Integration over $\mathbf{k}$ in Eqs. (19), (20) is carried out over the region $M + \cos\theta_s > 0$, which includes all the sound waves outgoing from the discontinuity, and integration over $\mathbf{k}_v$ and $\mathbf{k}_\sigma$, as in the previous formulas, over the regions $k_{vz} > 0$ and $k_{\sigma z} > 0$. The quantities $(A^2)_\mathbf{k}$ in (19) and (20) are equal to $\frac{1}{2}\rho T c^2$ for sound incident from the region $z < 0$ (all the quantities which refer to the supersonic portion $z < 0$ are denoted by the prime), c_p'/ρ' for incident entropy waves, and T'/ρ' for vortical waves. The derivatives

$$\partial k_z/\partial k_{sz}, \quad \partial k_z/\partial k_{vz} = \partial k_z/\partial k_{\sigma z}$$

are easily calculated if we note that they represent the ratio of the group velocities of the corresponding outgoing and incident waves.

We should add another term to the expression (20) for $\overline{(\delta v_x)^2} + \overline{(\delta v_y)^2}$, due to the contribution of vortical waves polarized perpendicular to the plane of incidence. Inasmuch as the transmission coefficient of such waves through the discontinuity is equal to unity, the corresponding contribution has a very simple form:

$$\overline{(\delta v_x)^2} + \overline{(\delta v_y)^2} = \int \frac{d^3\mathbf{k}_v}{(2\pi)^3}\left\{\frac{v}{v'}\frac{2T'}{\rho} - \frac{2T}{\rho}\right\}\exp(-2\varkappa_v z). \quad (21)$$

Since all the coefficients W are functions of the angles between the wave vector and the z axis and do not depend on the absolute value of $\mathbf{k}$, integration over $|\mathbf{k}|$ in Eqs. (19)–(21) is easily performed. As a result, all the mean-square fluctuations turn out to be equal to the product of $z^{-3/2}$ and complicated angular integrals of functions which contain the coefficients W. These integrals are conveniently represented as integrals with respect to the variable $s = \tan^2\theta$, where θ is the angle between the z axis and the wave vector of the entropy (or vortex) wave, which has the same value of frequency and projection of the wave axis on the plane $z = 0$ as any considered wave. All the other angular variables are easily expressed in terms of s.

It is only necessary to have the following in mind. By virtue of the supersonic character of the flow in the region $z < 0$, sound waves with arbitrary $\mathbf{k}$ are incident on the discontinuity. It is clear that in this case, the same outgoing waves arise as a result of incidence of two different sound waves, i.e., that the function $k_{sz}(\mathbf{k})$ for the sound in the region $z < 0$ is double-valued. In summation over A in Eqs. (19) and (20), it is necessary to carry out the summation also over the two branches of this function. Taking all this into account, we obtain the following formulas for the non-equilibrium parts of the mean-square fluctuations:

$$\overline{(\delta p)^2} = I_1 z^{-3/2}, \quad \overline{(\delta\sigma)^2} = J_1 z^{-3/2}, \quad \overline{\delta p\delta\sigma} = 0,$$
$$\overline{(\delta v_z)^2} = (I_2 + J_2) z^{-3/2},$$
$$\overline{(\delta v_x)^2} + \overline{(\delta v_y)^2} = \left\{\frac{I_1}{\rho^2 c^2} - I_2 + J_3 + \frac{T' - T}{8\rho}\left(\frac{v\rho}{2\pi\eta}\right)^{3/2}\right\} z^{-3/2},$$
$$\overline{\delta p\delta v_z} = I_3 z^{-3/2}, \quad \overline{\delta\sigma\delta v_z} = -J_4 z^{-3/2};$$

$$I_1 = \int\limits_0^{M^2/(1-M^2)} ds\, F_1(s)\left(-\frac{df_1}{ds}\right), \quad I_2 = \int\limits_0^{M^2/(1-M^2)} ds\, F_1(s)\left(-\frac{df_1}{ds}\right)\frac{f_1^2(s)}{\rho^2 c^2}$$
$$I_3 = \int\limits_0^{M^2/(1-M^2)} ds\, F_1(s)\left(-\frac{df_1}{ds}\right)\frac{f_1(s)}{\rho c};$$
$$J_1 = \frac{1}{2}\int\limits_0^\infty \frac{ds}{(1+s)^{3/2}} F_2(s), \quad J_2 = \frac{1}{2}\int\limits_0^\infty \frac{s\,ds}{(1+s)^{3/2}} F_3(s),$$
$$J_3 = \frac{1}{2}\int\limits_0^\infty \frac{ds}{(1+s)^{3/2}} F_3(s), \quad J_4 = \frac{1}{2}\int\limits_0^\infty \frac{s^{1/2}ds}{(1+s)^2} F_4(s);$$
$$F_1(s) = \frac{1}{16}\left(\frac{\rho c}{\pi\gamma}\right)^{3/2} c[M + f_1(s)]^{5/2}\left\{-\frac{\rho T c}{M + f_1(s)} + \frac{\rho T c}{M + f_2(s)}(\overline{W}_s^{(s)})^2\right.$$
$$\left. + \frac{\rho' T' c'}{M' + g_1(s)}(W_{s1}^{(s)})^2 + \frac{\rho' T' c'}{M' + g_2(s)}(W_{s2}^{(s)})^2 + \frac{2c_p'}{\rho v}(W_s^{(\sigma)})^2 + \frac{2T'}{\rho v}(W_s^{(v)})^2\right\},$$
$$F_2(s) = \frac{1}{16}\left(\frac{\rho v c_p}{2\pi\varkappa}\right)^{3/2}\left\{-\frac{2c_p}{\rho} + \frac{M}{M + f_2(s)}\rho T c^2 (\overline{W}_\sigma^{(s)})^2\right.$$
$$+ \rho' T' c' \frac{v}{M' + g_1(s)}(W_{\sigma 1}^{(s)})^2 + \rho' T' c' \frac{v}{M' + g_2(s)}(W_{\sigma 2}^{(s)})^2$$
$$\left. + \frac{2c_p'}{\rho'}\frac{v}{v'}(W_\sigma^{(\sigma)})^2 + \frac{v}{v'}\frac{2T'}{\rho'}(W_\sigma^{(v)})^2\right\},$$
$$F_3(s) = \frac{1}{16}\left(\frac{\rho v}{2\pi\eta}\right)^{3/2}\left\{-\frac{2T}{\rho} + \frac{M}{M + f_2(s)}\rho T c^2 (\overline{W}_v^{(s)})^2 + \frac{2c_p' v}{\rho' v'}(W_v^{(\sigma)})^2\right.$$

$$+\rho' T' c' v\left[\frac{1}{M'+g_1(s)}(W_{v1}^{(s)})^2+\frac{1}{M'+g_2(s)}(W_{v2}^{(s)})^2\right]+\frac{v}{v'}\frac{2T'}{\rho}(W_v^{(v)})^2\Big\},$$

$$F_4(s)=\frac{1}{16}\left(\frac{\rho v}{\pi\left(\eta+\frac{\varkappa}{c_p}\right)}\right)^{3/2}\Big\{\frac{M}{M+f_2(s)}\rho T c^2 \overline{W}_\sigma^{(s)}\overline{W}_v^{(s)}$$

$$+\rho' T' c' v\left[\frac{W_{\sigma1}^{(s)}W_{v1}^{(s)}}{M'+g_1(s)}+\frac{W_{\sigma2}^{(s)}W_{v2}^{(s)}}{M'+g_2(s)}\right]+\frac{2c_p' v}{\rho' v'}W_\sigma^{(\sigma)}W_v^{(\sigma)}+\frac{2T'}{\rho}W_\sigma^{(v)}W_v^{(v)}\Big\};$$

$$f_1(s)=\frac{-s+[M^2-(1-M^2)s]^{1/2}}{M(1+s)},\quad f_2(s)=\frac{-s-[M^2-(1-M^2)s]^{1/2}}{M(1+s)};$$

$$g_{1,2}(s)=\frac{-c'v's\pm v[v^2+sc'^2(M'^2-1)]^{1/2}}{v^2+v'^2 s} \qquad (22)$$

and the dependence of the coefficients W on s is given in Appendix 1.

Finally, we substitute (22) in Eqs. (5)—(7) and find the coordinate dependence of the hydrodynamic quantities:

$$\bar p(z)-p(\infty)=\frac{z^{-3/2}}{1-M^2}\Big\{(T-T')\frac{rv^2}{16\rho T}\left(\frac{\rho v}{2\pi\eta}\right)^{3/2}$$

$$-M^2\left[\frac{1}{c}\left(\frac{\partial c}{\partial p}\right)_\sigma+\frac{r}{2\rho^2 T}\right]I_1+v^2\left[\frac{1}{2}\left(\frac{\partial r}{\partial\sigma}\right)_p-r\left(\frac{r}{\rho}+\frac{1}{2c_p}\right)J_1\right.$$

$$-\rho J_2-\frac{rv^2}{2T}\left(J_2+J_3+\frac{I_1}{\rho^2c^2}+\frac{2I_3}{\rho v}-\frac{2T}{v}J_4\right)\Big\},$$

$$\bar\sigma(z)-\sigma(\infty)$$

$$=-z^{-3/2}\Big\{\frac{T'-T}{16\rho T}\left(\frac{\rho v}{2\pi\eta}\right)^{3/2}+\frac{I_1}{\rho^2c^2T}$$

$$+\left(\frac{r}{\rho}+\frac{1}{2c_p}\right)J_1+\frac{J_2+J_3}{2T}+\frac{I_3}{\rho v T}-\frac{J_4}{v}\Big\},$$

$$\bar v(z)-v(\infty)=\frac{p(\infty)-\bar p(z)}{\rho v}-z^{-3/2}\Big\{\frac{I_2+J_2}{v}+\frac{I_3}{\rho c^2}-\frac{r}{\rho}J_4\Big\}.$$

The coordinate dependence of the average density is also easily obtained by using (1) and the formulas written down above:

$$\bar\rho(z)-\rho(\infty)=\frac{1}{c^2}(\bar p(z)-p(\infty))+r(\bar\sigma(z)-\sigma(\infty))$$

$$+z^{-3/2}\Big\{\frac{1}{2}\left(\frac{\partial r}{\partial\sigma}\right)_p J_1-\frac{1}{c^3}\left(\frac{\partial c}{\partial p}\right)_\sigma I_1\Big\}.$$

In conclusion, we pause on the problem of the region of applicability of the results obtained. The fundamental limitation is connected with the fact that we have used the hydrodynamic approach and have neglected possible dispersion of the kinetic coefficients. Inasmuch as the frequencies ω of the fluctuations which make the principal contribution to the coordinate dependence of the hydrodynamic quantities at a distance z from the discontinuity are equal in order of magnitude to $\omega\sim(\rho c^3/\gamma z)^{1/2}$, it is clear that the distance z ought to be so great that dispersion of the kinetic coefficients should be absent for the frequencies shown and the hydrodynamic approach should be applicable. For gases, the latter condition means that z ought to be large in comparison with the free path length of the particles, and for liquids, in comparison with $\rho c^3/\gamma\omega_0^2$, where ω_0 is the frequency above which significant dispersion of the kinetic coefficients begins. For simple liquids, in which dispersion is lacking up to frequencies of the order of c/a (a is the interatomic distance), the only condition is the inequality $z \gg a$.

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APPENDIX 1

If a sound wave with amplitude δp is incident on a shock wave from the region $z > 0$, then there appear a reflected sound wave with amplitude $\overline{W}_s^{(s)}\delta p$, a reflected entropy wave with amplitude $\delta\sigma=\overline{W}_\sigma^{(s)}\delta p$ and a reflected vortical wave, polarized in the plane of incidence, with amplitude $v_1=\overline{W}_v^{(s)}\delta p$. The coefficients $W^{(s)}$ are functions of the angle of incidence. It is convenient, however, to choose another angular variable in place of the angle of incidence, namely $s=\tan^2\theta$, where θ is the angle between the wave vector of the reflected vortical (or entropy) wave and the z axis. We then have

$$\overline{W}_s^{(s)}=-h(s)\{1+Mf_2(s)-G(s)[1+M+2Mf_2(s)]\},$$

$$\overline{W}_v^{(s)}=h(s)\frac{1-M^2}{\rho v}\frac{2s^{1/2}}{(1+s)^{1/2}}\frac{(v-v')G(s)}{v+v's}[M^2-(1+M^2)s]^{1/2},$$

$$\overline{W}_\sigma^{(s)}=\frac{v}{T}\left(\frac{1+s}{s}\right)^{1/2}\overline{W}_v^{(s)};$$

$$h(s)=\{1+Mf_1(s)-G(s)[1+M^2+2Mf_1(s)]\}^{-1};$$

$$f_1(s)=\frac{-s+[M^2-(1-M^2)s]^{1/2}}{M(1+s)},\qquad f_2(s)=\frac{-s-[M^2-(1-M^2)s]^{1/2}}{M(1+s)},$$

$$G(s)=\frac{(v+v's)\rho T}{(v-v')\{rv^2(1+s)-2\rho Ts\}}.$$

For incidence of a sound wave from the supersonic region $z < 0$, the dependence of the angle of incidence on the variable s is double valued. The cosine of the angle of incidence is equal to

$$g_{1,2}(s)=\frac{-c'v's\pm v[v^2+sc'^2(M'^2-1)]^{1/2}}{v^2+v'^2s},$$

where the sign in front of the square root is chosen in correspondence with the direction of the wave vector of the incident wave.

The amplitudes of the sound wave $W_{s1,2}^{(s)}\delta p$, the entropy wave $W_{1,2}^{(s)}\delta p$ and the vortical wave $W_{v1,2}^{(s)}\delta p$, all originating in the region $z > 0$, are connected with the amplitude of the incident wave δp by the relation

$$W_{s1,2}^{(s)}=h(s)\{M'[M'+g_{1,2}(s)]-G(s)[1+M'^2+2M'g_{1,2}(s)]\},$$

$$W_{v1,2}^{(s)}=-h(s)\frac{(v-v')G(s)}{\rho v}\frac{(s(1+s))^{1/2}}{v's+v}\{M(M'^2-1)f_1(s)+M^2M'^2-1$$

$$-M'(1-M^2)g_{1,2}(s)\},\qquad W_{\sigma1,2}^{(s)}=\frac{v}{T}\left(\frac{1+s}{s}\right)^{1/2}W_{v1,2}^{(s)}.$$

The transformation coefficients of the vortical wave, polarized in the plane of incidence, incident from the region $z < 0$, and the other types of waves, are equal to

$$W_s^{(v)}=h(s)\rho'\left(\frac{v^2+v'^2s}{s}\right)^{1/2}\Big\{\frac{v'}{v-v'}-\frac{v'^2s}{v^2+v'^2s}$$

$$-G(s)\left(\frac{v^2r}{\rho'T}-\frac{2v'^2s}{v^2+v'^2s}\right)\Big\},$$

$$W_v^{(v)}=-h(s)\frac{v-v'}{v'(v+v's)}[(v^2+v'^2s)(1+s)]^{1/2}G(s)\Big\{\left(\frac{2v'^2s}{v^2+v'^2s}-\frac{v^2r}{\rho'T}\right)$$

$$\times[1+Mf_1(s)]+\left(\frac{v'}{v-v'}-\frac{v'^2s}{v^2+v'^2s}\right)[1+M^2+2Mf_1(s)]\Big\},$$

$$W_\sigma^{(v)}=-\frac{1}{T}\left(\frac{v^2+v'^2s}{s}\right)^{1/2}+\frac{v}{T}\left(\frac{1+s}{s}\right)^{1/2}W_v^{(v)}.$$

Similar formulas for the transformation coefficient of the entropy wave have the following form:

$$W_s^{(\sigma)}=h(s)v'^2r'\left[1-G(s)\left(1-\frac{v^2}{v'^2}\frac{r}{r'}\frac{T'}{T}\right)\right],$$

$$W_v^{(\sigma)}=-h(s)\left(\frac{s}{1+s}\right)^{1/2}\frac{(v-v')(1+s)}{v+v's}G(s)\frac{r'v'}{\rho}\Big\{M^2+\frac{v^2}{v'^2}\frac{r}{r'}\frac{T'}{T}$$

$$+\left(1+\frac{v^2}{v'^2}\frac{r}{r'}\frac{T'}{T}\right)Mf_1(s)\Big\},\qquad W_\sigma^{(\sigma)}=\frac{T'}{T}+\frac{v}{T}\left(\frac{1+s}{s}\right)^{1/2}W_v^{(\sigma)}.$$

APPENDIX 2

We start out from the general equation of entropy transfer in a liquid, written down with account of the external force $g_p(t)$ (see[9])

$$\partial\sigma_p/\partial t+(i\mathbf{p}\mathbf{v}+\chi p^2)\sigma_p=-g_p(t)/\rho T, \qquad (A.1)$$

where $\sigma_p(t)$ are the spatial Fourier components of the entropy, $\chi=\kappa/\rho c_p$ is the coefficient of temperature conductivity of the liquid. For the function $F(\mathbf{p},\mathbf{p}';t)=\overline{\sigma_p^*(t)\sigma_{p'}(t)}$ we easily obtain the following equation from (A.1)

$$\partial F(\mathbf{p},\mathbf{p}';t)/\partial t=i\mathbf{v}(\mathbf{p}-\mathbf{p}')F(\mathbf{p},\mathbf{p}';t)-\chi(p^2+p'^2)F(\mathbf{p},\mathbf{p}';t) - (\overline{g_p^*\sigma_{p'}}+\overline{\sigma_p^*g_{p'}})/\rho T. \qquad (A.2)$$

If the liquid is weakly inhomogeneous, then, setting $\mathbf{p}=-\mathbf{k}-\mathbf{q}/2$, $\mathbf{p}'=-\mathbf{k}+\mathbf{q}/2$, we have $q\ll k$ and Eq. (A.2) can be rewritten in the form

$$\partial F(\mathbf{k},\mathbf{q};t)/\partial t+(i\mathbf{q}\mathbf{v}+2\chi k^2)F(\mathbf{k},\mathbf{q};t) + (\overline{g_p^*\sigma_{p'}}+\overline{\sigma_p^*g_{p'}})/\rho T=0. \qquad (A.3)$$

The calculation of the last two terms in (A.3) can be carried out by assuming the liquid to be entirely homogeneous. In this case, as is seen from (A.1),

$$\sigma_p(\omega)=-\frac{i}{\rho T}\frac{g_p(\omega)}{\omega-i\chi p^2},$$

where $\sigma_p(\omega)$, $g_p(\omega)$ are the temporal Fourier coefficients of the entropy and the random force, and, according to Landau and Lifshitz,[9]

$$\overline{g_p^*(\omega)g_{p'}(\omega')}=2p^2T^2\varkappa(2\pi)^4\delta(\mathbf{p}-\mathbf{p}')\delta(\omega-\omega').$$

Using the last equation, we easily find

$$\overline{\sigma_p^*(t)g_{p'}(t)}+\overline{\sigma_{p'}(t)g_p^*(t)}=i\frac{2p^2T\varkappa}{\rho}(2\pi)^2 \times\delta(\mathbf{p}-\mathbf{p}')\left\{\int_{-\infty}^{\infty}\frac{d\omega}{\omega-i\chi p^2}-\int_{-\infty}^{\infty}\frac{d\omega}{\omega-i\chi p^2}\right\}=-\frac{2p^2T\varkappa}{\rho}(2\pi)^3\delta(\mathbf{p}-\mathbf{p}'). \qquad (A.4)$$

The mean-square fluctuations of the entropy are equal to

$$\overline{(\delta\sigma)^2}=\int\frac{d^3\mathbf{p}\,d^3\mathbf{p}'}{(2\pi)^6}F(\mathbf{p},\mathbf{p}';t)e^{i(\mathbf{p}'-\mathbf{p})\mathbf{r}}=\int_{k_z>0}\frac{d^3\mathbf{k}}{(2\pi)^3}f_\mathbf{k}(r,t), \qquad (A.5)$$

where we have introduced the fluctuation distribution function

$$f_\mathbf{k}(\mathbf{r},t)=\int\frac{d^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\mathbf{r}}\{F(\mathbf{k},\mathbf{q};t)+F(-\mathbf{k},\mathbf{q};t)\},$$

which, in accord with (A.3) and (A.4), satisfies the following kinetic equation

$$\partial f_\mathbf{k}/\partial t+\mathbf{v}\partial f_\mathbf{k}/\partial\mathbf{r}+2k^2\chi(f_\mathbf{k}-2c_p/\rho)=0.$$

In our case, the problem is stationary and all the quantities depend only on the coordinate z. We therefore have

$$v\partial f_\mathbf{k}/\partial z+2k^2\chi(f_\mathbf{k}-2c_p/\rho)=0. \qquad (A.6)$$

By comparing (A.5) with Eq. (18), we find that the function $f_{\mathbf{k}\sigma}$ at $z=0$ satisfies the boundary condition

$$f_{\mathbf{k}_\sigma}|_{z=0}=\sum_A\frac{\partial k_z}{\partial k_{\sigma z}}2|W_\sigma^{(A)}|^2(A^2)_\mathbf{k}.$$

The solution of Eq. (A.6) with such a boundary condition has the form

$$f_{\mathbf{k}_\sigma}(z)=\left\{\sum_A\frac{\partial k_z}{\partial k_{\sigma z}}2|W_\sigma^{(A)}|^2(A^2)_\mathbf{k}-\frac{2c_p}{\rho}\right\}\exp\left(-\frac{2k_\sigma^2\chi}{v}z\right),$$

which corresponds in accuracy with Eq. (19).

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