Internal friction in solid helium

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The dissipation of elastic energy in solid helium on nonuniform time-oscillating deformation of the crystal is studied theoretically. Delocalization of point defects due to the quantum nature of the helium crystals leads to a significant difference in the temperature dependence of the energy absorption coefficient from the corresponding dependence for localized defects in classical crystals. The forms of temperature and frequency dependence of the absorption coefficient in various temperature ranges are determined by the relation between the temperature and the bandwidth of the delocalized defects, by the defecton statistics, and by the magnitude and temperature dependence of the defecton-phonon relaxation time. It is shown that an inhomogeneous static deformation applied to the crystal gives rise to resonance absorption peaks at frequencies proportional to the gradient of static deformation. Expressions for evaluating the quality factor of concrete oscillation systems are given for all crystallic modifications of both pure helium isotopes and weak solutions.

The study of point defects in solid helium has recently acquired substantial importance due to quantum effects that become appreciable because of the large amplitude of zero-point vibrations in the helium lattice. At sufficiently low temperatures, the point defects (vacancies, impurities) are delocalized and must be regarded as quantum objects—defectons having a finite bandwidth Δ.[1]

The parameters of the defecton energy spectrum and the relaxation times have hitherto been determined from nuclear magnetic resonance experiments[2,3]. The information obtained in such experiments is difficult to interpret fully unambiguously because of the complex nature of relaxation processes in the diffusion of magnetic moments. In consequence, many important features of the defects (e.g., the bandwidths), determined from the NMR experiments, are known only qualitatively.

Internal friction provides a sensitive tool for studying the defect properties and relaxation processes in crystals. Such study can contribute substantially to our knowledge of the dynamic and kinetic properties of defects in solid helium, which are of fundamental interest, as well as provide a means for observing various quantum processes, such as the Bose condensation.

In the present paper, formulas are derived to describe the absorption of elastic energy in solid helium. The results obtained make it possible to interpret the experimental data on internal friction over the whole range of temperatures in which the crystalline phase of helium exists. Due to the quantum nature of solid helium, the temperature dependence of the elastic energy absorption coefficient is essentially different from that of a crystal with localized defects.

In the first section of the paper, the absorption coefficient is evaluated for the case when the delocalized defects are acted upon by a time-periodic volume force $f = f_0 \sin \omega t$.

In the second section the corresponding formulas for the localized defects are given for comparison.

In the third section it is shown that if a constant force F acts on the defectons along with the oscillating force f, then the maximum of the absorption curve at $\omega = 0$ breaks up into several high-frequency resonance peaks.

The possibility of creating a space-uniform volume force acting on the defects is discussed in the Appendix. It also contains formulas relating the value of power absorbed per unit volume to the Q of a particular oscillatory system.

Let us first say a few words about how we propose to study internal friction. As is known, a uniform deformation does not give rise to dissipation of elastic energy by vacancies and substitutional impurities having the same symmetry as the crystal lattice. The absorption of energy, involving diffusion currents of defects, comes into play under the action of a volume force proportional to the gradient of deformation[4].

The study of internal friction is usually concerned with observation of an extremum of Q as a function of the external-field frequency ω. The position of absorption peaks is governed by relations of the type $\omega \tau^* \sim 1$, where τ^* is a relaxation time. Examples of such relaxation times are the time τ to establish a diffusive particle current proportional to the disturbing force and the time τ_D, which describes the time required for the number of particles throughout the whole sample to reach equilibrium. Considering that vacancies and impurities can appear and disappear only on the crystal surface, τ_D is related to the diffusion coefficient D and the typical linear dimension of the sample L by the relation $\tau_D \sim L^2/D$. For thin samples the position of the absorption maximum $\omega \tau_D \sim 1$ is in a reasonable frequency range. It is this dimensional extremum of Q that is usually studied in Gorsky-effect experiments[5]. The time τ is determined by the relation $\tau \sim l^2/D$, where l is the mean free path of the defects. In classical crystals the mean free path l is equal to the interatomic distance a and the diffusion coefficient is typically the value $D \sim 10^{-4}–10^{-6}$ cm^2/sec, so one obtains an estimate of $\tau \sim 10^{-10}–10^{-12}$ sec. In consequence of this the resonance peak at frequencies $\omega \tau \sim 1$ is practically unobservable in classical crystals.

The situation is different in quantum crystals. According to Andreev and Lifshitz[1], as the temperature T decreases, the defecton mean free path starts to exceed the interatomic distance and τ increases as T^{-9}. Experimental observation of the absorption maximum at the frequencies $\omega \tau \sim 1$ becomes possible in the new situation. Due to the large value of the defecton mean free path in the helium crystal, the structure of the ab-

sorption peak at $\omega\tau_D \sim 1$ is sensitive to the details of sample surface structure and shape. The picture of internal friction becomes more complicated, and at $l \sim L$ a kind of size effect may appear. In the present work we studied, for the sake of simplicity, purely volume internal friction under the conditions $\omega\tau \sim 1$ and $\omega\tau_D \gg 1$.

In order to meet the last condition the inequality $\omega\tau L^2/l^2 \gg 1$ must be satisfied. On the other hand, the oscillation wavelength must be assumed to be much larger than the sample size L to avoid spatial dispersion of the volume force acting upon the defect. Both inequalities hold consistently if

$$L \gg \frac{\Delta^2 a^2}{\hbar c \Theta}\left(\frac{\Theta}{T}\right)^9, \quad \frac{c}{L} \gg \omega \gg \frac{\Delta^2 a^2}{L^2 \hbar \Theta}\left(\frac{\Theta}{T}\right)^9,$$

where Θ is the Debye temperature of solid helium, Δ is the defecton bandwidth, and c is the propagation velocity of the corresponding elastic vibrations, which is of the same order of magnitude as the sound velocity. In what follows we shall assume that the above conditions are satisfied.

1. In order to evaluate the internal friction it is necessary to solve the kinetic equation for the defecton density matrix $\hat{n}$. Because of the spatial uniformity of the external force f, the classical kinetic equation gives exactly the same result for the distribution function n as the corresponding quantum equation. It is permissible to use the τ-approximation in our problem. This approximation is justified for defecton-phonon scattering provided the bandwidth Δ is much smaller than the characteristic phonon energy T. It holds also in the opposite limiting case, since for $\Delta \gg T$ the defecton energy spectrum can be considered isotropic, as will be shown below.

On the basis of the previous discussion the starting kinetic equation can be written in the form:

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon(\mathbf{p})}{\partial \mathbf{p}} + \frac{\partial n}{\partial \mathbf{p}}f_0 \sin\omega t + \frac{n-n_0}{\tau} = 0, \tag{1}$$

where n_0 is the equilibrium defecton energy distribution function, $\varepsilon(\mathbf{p})$ is the defecton energy spectrum, and $\mathbf{p}$ is the quasimomentum. The power W absorbed per unit volume is given by the ensemble-and-time average of the expression $(\hat{\mathbf{v}}\hat{f} + \hat{f}\hat{\mathbf{v}})/2$ ($\hat{\mathbf{v}} \equiv \partial\varepsilon/\partial\mathbf{p}$ is the velocity operator). In our case

$$W = \left\langle \int \frac{d^3p}{(2\pi\hbar)^3}n f_0 \sin\omega t \frac{\partial\varepsilon}{\partial\mathbf{p}}\right\rangle, \tag{2}$$

where the angle brackets denote time averaging, and integration is performed over the reciprocal lattice cell.

In solving (1) for a small value of the external force $f_0 \ll \hbar\omega/a$ (or $f_0 \ll \hbar\omega T/a\Delta$ if $\Delta \ll T$), we can use an approximation linear in f_0. Let $n = n_0 + n_1 + o(t)$, where

$$n_1 = -\frac{\partial n_0}{\partial\varepsilon}\frac{\partial\varepsilon}{\partial\mathbf{p}}f_0\tau\frac{\sin\omega t - \omega\tau\cos\omega t}{1+\omega^2\tau^2},$$

and $o(t)$ tends to zero as $\exp(-t/\tau)$. Then (2) assumes the form

$$W = -\frac{\tau}{2(1+\omega^2\tau^2)}\int \frac{d^3p}{(2\pi\hbar)^3}\frac{\partial n_0}{\partial\varepsilon}\left(f_0\frac{\partial\varepsilon}{\partial\mathbf{p}}\right)^2. \tag{3}$$

We shall first determine the absorption coefficient W in the temperature range where the condition $T \gg \Delta$ holds. In this case the band is populated uniformly, and the shape of the energy spectrum $\varepsilon(\mathbf{p})$ can be determined by using the tight-binding approximation. There are three possible types of crystal lattice for both He⁴

and He³ (the phase diagrams of both isotopes are given, for instance, in the review[6]): body-centered cubic (bcc), face-centered cubic (fcc), and hexagonal close-packed (hcp). The energy spectra for these three phases in the tight-binding approximation are of the form:

$$\varepsilon(\mathbf{p}) = \varepsilon_0 + \frac{\Delta}{2} - \frac{\Delta}{2}\cos\frac{p_1 a}{\hbar}\cos\frac{p_2 a}{\hbar}\cos\frac{p_3 a}{\hbar} \tag{4a}$$

for the bcc lattice with translation vectors $(-a, a, a)$, $(a, -a, a)$, and $(a, a, -a)$;

$$\varepsilon(\mathbf{p}) = \varepsilon_0 + \frac{3}{4}\Delta - \frac{\Delta}{4}\left\{\cos\frac{p_1 a}{\hbar}\cos\frac{p_2 a}{\hbar} + \cos\frac{p_2 a}{\hbar}\cos\frac{p_3 a}{\hbar} + \cos\frac{p_1 a}{\hbar}\cos\frac{p_3 a}{\hbar}\right\} \tag{4b}$$

for the fcc lattice with translation vectors $(0, a, a)$, $(a, 0, a)$, and $(a, a, 0)$; and

$$\varepsilon(\mathbf{p}) = \varepsilon_0 + \frac{3}{4}\Delta - \frac{\Delta}{8}\left\{\cos\frac{p_1 a}{\hbar} + 2\cos\frac{p_1 a}{2\hbar}\cos\frac{\sqrt{3}p_2 a}{2\hbar}\right.$$
$$\left. \pm \cos\frac{\sqrt{2}p_3 a}{\sqrt{3}\hbar}\left[2\cos\frac{p_1 a}{\hbar} + 4\cos\frac{p_1 a}{2\hbar}\cos\frac{\sqrt{3}p_2 a}{2\hbar} + 3\right]^{1/2}\right\} \tag{4c}$$

for the hcp lattice with translation vectors $(a, 0, 0)$, $(a/2, \sqrt{3}a/2, 0)$, and $(0, 0, \sqrt{8}a/\sqrt{3})$, and two sublattices separated by the vector $(0, -a/\sqrt{3}, \sqrt{2}a/\sqrt{3})$.

The constants in (4) are chosen so as to make the bandwidth for each lattice equal to Δ and the energy gap to ε_0. It might be well to emphasize that in the hcp case both branches of the spectrum (4c) should be taken into account in the integration over the momenta. The method of evaluating the spectrum in the tight-binding approximation, as usually described in the literature, is often based on the book by Sommerfeld and Bethe[7]. However, one has to realize that the relevant expression ([7], p. 397) describing the energy spectrum for a crystal with several atoms per unit cell (e.g. the hcp lattice) does not allow for energy degeneracy, and therefore yields an incorrect spectrum that has only one branch and is aperiodic in the reciprocal lattice translation vectors. The review by Reitz[8] also contains a spectrum which is aperiodic in the reciprocal lattice of an hcp crystal. The energy spectrum of the hcp lattice (4c) can be obtained, for instance, but solving the secular equation for an energy matrix whose elements are the three- or two-center integrals listed in[9].

The power W absorbed per unit volume is given by (3). Using the spectra (4), W can be readily evaluated. For both cubic phases we obtain

$$W = \frac{1}{64}N\frac{\Delta^2}{T}\frac{f_0^2 a^2}{\hbar^2}\frac{\tau}{1+\omega^2\tau^2}[1\mp\nu], \tag{5a}$$

where N is the number of defects per unit volume, ν is the atomic concentration of the defects (number of particles per atom), and the $+$ and $-$ signs refer to the cases of Bose and Fermi statistics, respectively. The absorption coefficient in the hcp lattice is equal to

$$W = \frac{1}{64}N\frac{\Delta^2}{T}\frac{a^2}{\hbar^2}(\alpha f_\perp^2 + f_\parallel^2)\frac{\tau}{1+\omega^2\tau^2}[1\mp\nu], \quad \alpha = 1 - \frac{3\sqrt{3}}{16\pi}, \tag{5b}$$

where $f_\parallel$ and $f_\perp$ are, respectively, the projections of the force f_0 on the hexagonal axis $(0, 0, 1)$ and on a plane perpendicular to it. The defecton concentration is usually assumed to be small, and the term in the square brackets in (5) can be neglected unless the effects quadratic in concentration are taken into account.

The condition $T \gg \Delta$ can hold for both vacancies and impurities. In the case of impurities the concentration N can be regarded as a constant independent of temperature. Therefore, the temperature dependence of

A. É. Meĭerovich

the absorption coefficient for impuritons is given by the factor $\tau/T(1 + \omega^2\tau^2)$. For $\omega\tau \ll 1$ the magnitude of W behaves like $W \sim \tau/T \sim T^{-10}$. If we fix the frequency ω and lower the temperature, then the condition

$$1/\omega\tau \approx (T/\Theta)^9 (c/a\omega) \ll 1$$

will hold for sufficiently low temperatures, and the behavior of W will be given by $W \sim 1/T\tau \sim T^8$. Thus, with temperature descreasing in the region $T \gg \Delta$, the absorption coefficient first increases as T^{-10} and then, starting from a certain temperature, decreases as T^8. The maximum of the absorption curve as a function of temperature at a fixed frequency corresponds to the value of T at which $\omega\tau \approx 1$. This temperature behavior of the absorption curve is essentially distinct from the classical-diffusion behavior of W, in which W tends exponentially to zero as the temperature decreases. In the case of vacancions with a comparatively large energy gap ϵ_0 (which corresponds to the activation energy of vacancies in the classical case), the concentration N depends on T as $\exp(\sim\epsilon_0/T)$, and the behavior of the W(T) curve differs by this exponential factor from the corresponding dependence for impuritons.

Now let us investigate the reverse limiting case, $T \ll \Delta$. We note that under this condition it is reasonable to consider only the vacancions with $\Delta \sim 1K$ ($\Delta \sim 10^{-6}$ K for the case of impuritons). The inequality $T \ll \Delta$ means that all the particles are near the bottom of the band. In this case it suffices to assume a quadratic dependence of the energy on the quasimomentum in the vicinity of the minimum of $\epsilon(p)$:

$$\varepsilon(p) = \varepsilon_0 + p^2/2m, \tag{6a}$$

and to extend the integration in (3) to the whole momentum space. The effective-mass tensor in (6a) is isotropic due to the spherical symmetry of the minima of the functions $\epsilon(p)$ in (4). The spectra (4), obtained in the tight-binding approximation, cannot, in general, guarantee an accurate description of the behavior of $\epsilon(p)$ near its extrema. Nevertheless, on comparing (4) and (6a) one can infer with a certain accuracy that for all three lattice types

$$m \approx 2\hbar^2/a^2\Delta. \tag{6b}$$

If the possibility of existence of zero-energy vacancions is neglected, then one can assume that $\epsilon_0 \gg T$ for the whole temperature range $T \ll \Delta$ for vacancies. In this case the equilibrium distribution function takes the Boltzmann form $n_0 = \exp\{-\epsilon(p)/T\}$ with the energy spectrum $\epsilon(p)$ determined by the relations (6). A calculation yields

$$W = \frac{1}{2(2\pi\hbar^2)^{3/2}} e^{-\epsilon_0/T} T^{3/2} f_0^2 m^{3/2} \frac{\tau}{1+\omega^2\tau^2} = N\frac{f_0^2}{2m}\frac{\tau}{1+\omega^2\tau^2} \tag{7a}$$

or, employing (6b),

$$W = N\Delta\frac{f_0^2 a^2}{4\hbar^2}\frac{\tau}{1+\omega^2\tau^2}, \qquad N = \frac{e^{-\epsilon_0/T}}{\pi^{3/2}a^3}\left(\frac{T}{\Delta}\right)^{3/2}. \tag{7b}$$

In all of the above it was assumed that the equilibrium defecton concentration in the absence of the external force f was determined by minimizing the energy as a function of the number of particles. This condition is equivalent to requiring the chemical potential to be zero. However, it was mentioned in the beginning of the article that the defectons can appear and disappear only on the sample surface, and hence the time required to establish the equilibrium with respect to the number of particles is macroscopically large. It is

therefore interesting to study the behavior of the absorption coefficient not only under the above assumption that the vacancy concentration N is defined from the conditions of thermal equilibrium with respect to the number of particles, but also for the case when the concentration N is given by a constant independent of temperature. In this case the behavior of W(T) depends essentially on the type of statistics for vacancions.

For the case of Fermi statistics (e.g., the vacancies in He^3), provided the number of particles is small, $(Na^3)^{2/3} \ll 1$ (this condition can always be assumed satisfied), the Fermi energy ϵ_F lies near the bottom of the band and $(\epsilon_F - \epsilon_0)/\Delta \ll 1$. As a result for $\Delta \gg T$ the equilibrium distribution function is of the form $\partial n_0/\partial\epsilon = -\delta(\epsilon_F - \epsilon)$ and the spectrum (6) can be employed as $\epsilon(p)$. As a result we obtain the following value for W:

$$W = N\frac{f_0^2}{2m}\frac{\tau}{1+\omega^2\tau^2} = N\Delta\frac{f_0^2 a^2}{4\hbar^2}\frac{\tau}{1+\omega^2\tau^2}. \tag{8}$$

For bosons (the vacancies in He^4) at temperatures below the Bose-condensation temperature

$$T_0 = \frac{1}{2}\left\{\frac{4\pi^2}{\Gamma(3/2)\zeta(3/2)}\right\}^{2/3}(Na^3)^{2/3}\Delta,$$

the absorption coefficient is of the form:

$$W = N\frac{f_0^2}{2m}\left(\frac{T}{T_0}\right)^{3/2}\frac{\tau}{1+\omega^2\tau^2} = N\Delta\frac{f_0^2 a^2}{4\hbar^2}\left(\frac{T}{T_0}\right)^{3/2}\frac{\tau}{1+\omega^2\tau^2}. \tag{9}$$

In this case the absorption curve W(T) has a kink at $T = T_0$.

The discussion of (5) can be applied with practically no modification to the temperature dependence W(T) in the case $\Delta \gg T$. As $T \to 0$, the coefficient W given by (7) tends exponentially to zero. The W(T) curves given by (8) and (9) have their maxima at $\omega\tau \approx 1$ for a fixed frequency ω. The relations (5) and (7)—(9) describe the behavior of W(T) for delocalized defects in the whole temperature range. It might be well to note that for $T = const$ the $W(\omega)$ curve has a maximum at $\omega \approx 0$.

2. In this section we shall consider the absorption of elastic energy in the case when the force f acts upon localized defects.

As the temperature increases, the defect-phonon relaxation time τ becomes shorter than the time that a defect spends at a lattice site. As a consequence the point defects become localized and perform diffusive random walks over the lattice sites. The kinetic equation (1), which proves inapplicable in this case, must be replaced by a system of equations in $n_i(t)$ (the probability for a localized defect to be at the i-th lattice site at the time t):

$$\frac{\partial n_i}{\partial t} = \sum_k (q_{ik}n_k - q_{ki}n_i).$$

The matrix element q_{ik} depends on the force applied to the defect, and defines the probability per unit time for the defect to migrate from site k to site i. It is sufficient to confine the summation over k to the nearest neighbors of i. In the approximation linear in the force, the coefficients q_{ik} can be expanded in a series in f, taking only the first terms:

$$q_{ik} = q_{ik}^{(0)} + (\delta q_{ik})f,$$

where $q_{ik}^{(0)}$ is the hopping probability in the unperturbed lattice.

The above expressions conform to the usual relation between the particle current and the external force

$$j = N\hat{B}f,$$

where we have introduced the mobility tensor $\hat{B}$ and the defect number density N. If the force f is constant in time, the corresponding tensor $\hat{B}_c$ is then related to the usual diffusion tensor $\hat{D}$ by the Einstein formula $\hat{B}_c = \hat{D}/T$. The relation between the mobility tensor $\hat{B}_s$ for a time-oscillating force and the tensor $\hat{B}_c$ for a constant force is governed by the relation between $(\delta_{qik})_s$ in a periodic field and $(\delta_{qik})_c$ in a constant field.

As is known, the relaxation process in a stochastic system is described by the Fokker-Planck equation. On averaging over energy, the Fokker-Planck equation transforms in the hydrodynamic limit into the well-known Langevin equation (see, e.g.[10]). The velocity of a Brownian particle, given by the Langevin equation, in our case corresponds precisely to the transition probability q_{ik}. It readily follows from the Langevin equation that

$$\text{Re}\,\hat{B}_s = \hat{B}_c/(1+\omega^2\tau^2) = \hat{D}/T(1+\omega^2\tau^2).$$

The power absorbed per unit volume is given by the time average of $f \cdot j$. For an oscillating force $f = f_0 \cos \omega t$ we get

$$W = \langle fj \rangle = N \langle (f\hat{B}_s f) \rangle = \frac{N}{2T}\frac{(f_0 D f_0)}{1+\omega^2\tau^2}. \tag{10a}$$

If the diffusion coefficient is isotropic, then

$$W = N D f_0^2/2T(1+\omega^2\tau^2). \tag{10b}$$

For localized defects the quasiequilibrium condition $\omega\tau \ll 1$ is usually assumed to be satisfied and the dispersion factor $(\omega^2\tau^2+1)^{-1}$ is disregarded.

The temperature dependence of the absorption coefficient is determined by the expression ND/T. As in (5), N is assumed to be temperature-independent for impurities, and for vacancies with temperature decreasing the density falls off exponentially. In the high-temperature region the diffusion coefficient obeys the exponential Arrhenius law $D \sim \exp(-U/T)$, where the potential U corresponds to the height of the barrier which a particle has to overcome in order to make a transition to a neighboring site. Thus, for large T the quantity W/N behaves like $\exp(-U/T)/T$.

As the temperature decreases, the probability of thermally activated hopping in quantum crystals becomes smaller than the probability of quantum tunneling (for more details see[11]). Quantum diffusion of delocalized defects occurs at such temperatures. In this case the coefficient D does not depend on temperature, and W/N behaves as 1/T. Introducing a quantum tunneling probability $1/\tau_0$ related to D by $D \sim a^2\Delta^2\tau_0/\hbar^2$, we can write expression (10b) in a form similar to (5):

$$W = N \frac{\Delta^2}{T}\frac{f_0^2 a^2}{\hbar^2}\frac{\tau_0}{1+\omega^2\tau^2}. \tag{11}$$

Formulas (10) and (11) determine the energy absorption in the whole range of temperatures above the delocalization temperature $T \sim \Theta(\Delta/\Theta)^{1/9}$.

3. It was pointed out above that the absorption coefficient W is a monotonic function of the external-field frequency ω. The reason for this is that we have chosen a defecton with a quasiclassical spectrum $\epsilon(p)$ as the unperturbed particle, and the disturbing force was spatially uniform. If the defecton had a discrete spectrum the absorption would then have maxima at the transition frequencies. Such absorption peaks can be obtained experimentally as follows.

Indeed, if even a very weak constant force F (e.g., a nonuniform static strain) is applied to a delocalized particle with the spectrum $\epsilon(p)$, then, due to the finite bandwidth Δ, the particle is bound by this potential, becomes localized, and acquires discrete energy levels. Such a localized state extends over a distance of order Δ/F, and the transition frequencies between the levels of the discrete energy spectrum $\omega \simeq Fa/\hbar$. As a consequence, the energy absorption that occurs when an oscillating force $f_0 \cos \omega t$ acts upon the particles along with the constant force F must have extrema at the frequencies of order $Fa/\hbar$. The corresponding W will be evaluated below.

Consider an unperturbed Hamiltonian of the form:

$$\hat{H}_0 = E_0 + \hat{\epsilon}(p) - F\hat{r}. \tag{12}$$

For simplicity we shall restrict ourselves to the bcc lattice. According to (4a), the spectrum $\hat{\epsilon}(p)$ can be written in the form

$$\hat{\epsilon}(p) = \frac{\Delta}{8}\sum_l \cos\left(\frac{a}{\hbar}l\hat{p}\right), \tag{13}$$

where summation is performed over the four positions of the vector l, namely (1, 1, 1), (−1, 1, 1), (1, −1, 1), and (1, 1, −1). The Schrödinger equation with the Hamiltonian (12) and (13) and periodic boundary conditions in the reciprocal lattice for the wave function ψ in p-representation can be readily solved. We obtain the following results for the eigenvalues and eigenfunctions of energy:

$$\epsilon_n = E_0 + \hbar\omega_n, \quad \omega_n = \frac{Fa}{\hbar}n\gamma_1, \quad \gamma_1 = (\gamma_{11}, \gamma_{12}, \gamma_{13}),$$

$$\psi_n(p) = \left(\frac{a}{2\pi\hbar}\right)^{\eta_1}\exp\left\{-\frac{i}{F\hbar}\int \epsilon(p)d(p\xi_1) + \frac{ia}{\hbar}pn\right\}. \tag{14}$$

The vector n here stands for the three quantum numbers (n, n_2, and n_3); the direction of ξ_1 is chosen along the force vector F. The matrix β_{ik} relates the coordinates x_i (in which the bcc translation vectors are of the form (4a)) to the system of coordinates ξ_i by the expression $\xi_i = \beta_{ik}x_k$. Any two mutually perpendicular axes in the plane normal to F can be chosen as ξ_2 and ξ_3. Let us also introduce the matrix γ_{ik} inverse to β_{ik}. In the following we shall be interested in the region of weak external fields $Fa \ll T$.

Let us now place the particles described by the Hamiltonian (12) in a time-oscillating external field $H = -f_0 \cdot r \cos \omega t$. To determine the absorption coefficient we must solve the kinetic equation for the density matrix $\hat{n}$:

$$\frac{\partial \hat{n}}{\partial t} + \frac{i}{\hbar}[\hat{H}_0 + \hat{H}_1, \hat{n}] + \frac{\hat{n} - \hat{n}_0}{\tau} = 0, \tag{15}$$

where $\hat{n}_0$ is the equilibrium density matrix which is diagonal in the states (14) of $\hat{H}_0$. The kinetic equation (15) can be linearized in f provided either of the following two systems of inequalities holds:

$$Fa \ll \hbar/\tau, \quad \frac{\Delta}{T}\frac{f_0 a}{\hbar/\tau}\frac{1}{(1+\omega^2\tau^2)} \ll 1,$$

$$Fa \gg \hbar/\tau, \hbar\omega; \quad f_0/F \ll T/\Delta.$$

We then obtain

$$\hat{n} = \hat{n}_0 + \hat{n}_1 \cos \omega t + \hat{n}_2 \sin \omega t,$$

$$(\hat{n}_1)_{k',k} = \frac{(n_{k'}-n_k)(\epsilon_{k'}-\epsilon_k-i\hbar/\tau)}{[\epsilon_{k'}-\epsilon_k-i\hbar/\tau]^2-\omega^2}(f\hat{r})_{k',k}. \tag{16}$$

The matrix elements in (16) are evaluated with the wave functions (14), n_k being the eigenvalues of the equilibrium density matrix $\hat{n}_0$ in the states (14).

The matrix elements of the operator $\hat{r}$ in the right-hand side of (16) are equal to

$$(\hat{r})_{\kappa',\kappa} = i\hbar\left(\frac{\partial}{\partial p}\right)_{\kappa',\kappa} = ak\delta_{\kappa'\kappa} - \frac{1}{32}\frac{\Delta}{F}\sum_l\frac{1}{(l\gamma_l)}\{\delta_{\kappa',\kappa+1}+\delta_{\kappa',\kappa-1}\},$$

where δ_{nm} is Kroneker's delta-function of the vector arguments $n = (n_1, n_2, n_3)$ and $m = (m_1, m_2, m_3)$.

The time-average power absorbed per unit volume is determined by the expression $W = \langle \text{Tr}(\hat{n}f \cdot v)\rangle$, where the velocity operator $\hat{v}$ is given by its matrix elements

$$(\hat{v})_{\kappa',\kappa} = \frac{i}{\hbar}(\varepsilon_{\kappa'}-\varepsilon_\kappa)(\hat{r})_{\kappa',\kappa}.$$

After straightforward but rather lengthy calculations we obtain the following value for the absorption coefficient:

$$W = N\frac{\Delta^2}{256T}\frac{a^2}{\hbar^2}\sum_l\frac{(f_0l)^2}{\tau}\frac{\omega_l^2+\omega^2+1/\tau^2}{[(\omega_l-\omega)^2+\tau^{-2}][(\omega_l+\omega)^2+\tau^{-2}]}. \quad (17)$$

It is apparent that W has its maxima at the frequencies $\omega \sim \omega_l \sim Fa/\hbar$ with the peak width $\delta\omega \sim 1/\tau$. As $F \to 0$, the expression (17) coincides with (5a).

On examination of (17), the following remarks can be made. First, all the results obtained in the first section suggest that the value of W as a function of the external-field frequency is maximal at $\omega = 0$. However, as is apparent from (17), any occasional nonuniform deformation of the crystal leads to a smearing and substantial anisotropy of such an absorption maximum. This attests to the fact that the precision of measurement of the internal friction increases with an increase of the oscillation frequency.

Secondly, as was pointed out above, the maximal absorption of elastic energy in the absence of the external force F occurs when $\omega \to 0$, whereas at high frequencies W behaves as $1/\omega^2$. Still, it is often more convenient to carry out the measurements at high frequencies. The expression (17) shows that one can shift the absorption peak into a higher frequency domain by creating a permanent nonuniform deformation of the crystal.

All the results obtained in this section refer, of course, to the temperature range in which the defects are delocalized. It makes sense to talk about bound states of the defectons in the field $-F \cdot r$ only provided the mean free path $l \sim \Delta a\tau/\hbar$ greatly exceeds the extension of the bound state Δ/F.

In conclusion, we note that the expressions for the value of power absorbed per unit volume obtained in the present study (Eqs. (5), (7), (11), and (17)) enable us to interpret unambiguously the experimental results on the internal friction in solid helium for a wide temperature range. Study of the frequency dependence $W(\omega)$ at different temperatures makes it possible to determine the temperature dependence of relaxation parameter $\tau(T)$. Knowing the behavior of $\tau(T)$ and the defect concentration N, one can easily determine the bandwidth Δ and the diffusion coefficient D.

The exponent in the $W(T)$ dependence for vacancions defines the energy gap ϵ_0.

It might be well to emphasize again that the temperature dependences of W differ essentially for localized and delocalized defects. For the delocalized defects, moreover, the function $W(T)$ behaves differently in the regions $\Delta \gg T$ and $\Delta \ll T$.

APPENDIX

A volume elastic force acting on the defects arises only on a nonuniform deformation of the crystal. According to[11],

$$f = K\Omega_{ik}\nabla u_{ik},$$

where u_{ik} is the deformation tensor, K^{-1} is the bulk modulus, and Ω_{ik} is a tensor having the dimensions of volume and characterizing the anisotropies of both the lattice and the defect. The convolution $\Omega_{ll}/3$ is equal to the change of the crystal volume brought about by introducing a defect; this quantity may be referred to as the volume per one defect Ω. For vacancies and substitutional impurities in high-symmetry crystals one can assume with some accuracy that

$$\Omega_{ik} = \Omega\delta_{ik}.$$

The force f proves to be spatially uniform if the elements of the deformation tensor are linear functions of the coordinates. The deformation tensor is known to have this form for bending vibrations of a rod. Expressions will be derived below which relate the values of the absorption coefficient W to the quality factor Q^{-1} of such vibrations.

Let us direct the axis ξ_1 along the bending direction of the rod, ξ_2 along the rod's axis, and ξ_3 so as to complete the Cartesian frame ξ_i. The transformation from the coordinates ξ_i to the coordinates x_i connected with the symmetry axes of the crystal lattice is performed by the matrix γ_{ik}: $x_i = \gamma_{ik}\xi_k$.

The deformation tensor in the case of bending vibrations of a rod is defined as (see[12]):

$$u_{ik} = \frac{E}{R}\beta_{iklm}\gamma_{l2}\gamma_{m2}\gamma_{p1}x_p,$$

where E is the elastic modulus of solid helium along ξ_2, R is the radius of curvature for bending, and β_{iklm} is the tensor inverse to the usual elastic tensor λ_{iklm} and evaluated in the frame x_i.

As the radius of curvature of the rod varies proportionally to $\cos\omega t$ from the value $1/R_0$ to $1/R = 0$, the force f acting on the defects is equal to

$$f_j = K\frac{E}{R}\Omega_{ik}\beta_{iklm}\gamma_{l2}\gamma_{m2}\gamma_{jl}\cos\omega t. \quad (A.1)$$

For a crystal of cubic symmetry with three nonzero elements of the elastic tensor $c_1 = \lambda_{iiii}$, $c_2 = \lambda_{iikk}$, and $c_3 = \lambda_{ikik}(i \neq k)$, we have (see[12]):

$$\frac{1}{E} = \frac{c_1+c_2}{(c_1-c_2)(c_1+2c_2)} + \left(\frac{1}{c_2}-\frac{2}{c_1-c_2}\right)(n_1^2n_2^2+n_1^2n_3^2+n_2^2n_3^2),$$

and the nonvanishing components of the tensor β_{iklm} are equal to

$$\beta_{iiii} = \frac{c_1+c_2}{(c_1-c_2)(c_1+2c_2)}, \quad \beta_{iikk} = -\frac{c_2}{(c_1-c_2)(c_1+2c_2)}, \quad \beta_{ikik} = \frac{1}{4c_3},$$

where $i \neq k$; n_i are the direction cosines of the ξ_2 axis in the x_i frame.

The above relations show the difficulties involved in employing the expression (A.1) even for a cubic crystal. Neglecting the anisotropy of the crystal, (A.1) can be rewritten in the form

$$f_j = \frac{E}{3R_0}\Omega\gamma_{jl}\cos\omega t. \quad (A.2)$$

A. É. Meĭerovich

Here E is the ordinary elastic modulus for an isotropic body.

The internal friction is usually studied by measuring the quality factor Q^{-1} rather than the magnitude of the power absorbed. For bending vibrations of a rod

$$Q^{-1} = \left(\frac{2\pi}{\omega} S \Big/ \frac{EI}{2R_0^2}\right) W, \qquad (A.3)$$

where S is the cross-sectional area of the rod and I is its moment of inertia about the ξ_3 axis. As one would expect, Q^{-1} does not depend on the vibration amplitude $1/R_0$, since W is quadratic in f.

Formulas (A.1)—(A.3), along with the formulas obtained for W in the present work, express Q^{-1} in terms of the properties of the defects, the elastic constants of solid helium, and the geometry of the sample. Values of the elastic constants of solid helium are given in the review[13].

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86