

Boundary Slip in Spin-Polarized Quantum Systems

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We describe the effects of boundary slip in spin-polarized quantum liquids and gases. The slip coefficients in boundary conditions form a 3×3 matrix. The off-diagonal coefficients are expressed via each other with the help of the Onsager relations. We calculate accurate lower and upper bounds of all slip coefficients for polarized degenerate Fermi liquids and for dilute gases at arbitrary temperatures. The calculations are based on the transport equation for spin-polarized systems with diffuse boundary conditions. The results for gases are especially simple in the limiting cases of low-temperature degenerate systems or in the high-temperature classical Boltzmann regime. All slip coefficients are proportional to the mean free path and increase with increasing spin polarization. As a by-product the theory describes the slip effects in binary mixtures of classical gases or Fermi liquids when the role of spin polarization is played by the concentration of the mixture.

1. INTRODUCTION

Physics of spin-polarized quantum systems is a rapidly developing branch of condensed matter physics at the intersection of atomic and low temperature physics. As always, such a fast development has its own drawbacks. Though we now understand, at least qualitatively, the vast majority of quantum effects in spin-polarized systems (see, e.g., reviews¹⁻⁹), some crucial parts of the overall picture are still missing.

The most important of these missing parts is the systematic description of boundary effects. While the bulk properties are studied in detail both

experimentally and theoretically, the boundary processes are touched only occasionally. Without a proper understanding of the boundary processes it is unrealistic to expect reliable results especially at very low temperatures when a rapid increase in mean free path makes the interaction with boundaries more and more crucial.

In this paper we focus on an important class of boundary phenomena, namely, on the boundary slip. This choice is not accidental, and is dictated by current experimental and theoretical needs. For technical reasons,^{1, 4, 8, 10} the experimental cell with polarized helium or hydrogen is often connected by a long diffusion channel with a chamber with different temperature and/or magnetization. These large gradients cause spin, thermal and mass flows. Though the source of these diffusion currents seems to be rather technical, such currents are practically unavoidable since the gradients always exist in experiments with systems in long-lived quasi-equilibrium polarized states. In the case of large gradients even a seemingly negligible boundary slip results in large mass, heat and spin flows through boundary layers. The thickness of these boundary layers is determined by the slip length ξ , and is proportional to the mean free path l . Since the mean free path and, therefore, the thickness of boundary layers in quantum gases increase dramatically with spin polarization,^{9, 11} the accurate interpretation of the existing experiments should be based on the use of slip boundary conditions.

From a theoretical point of view, the study of boundary slip in spin-polarized quantum systems has an additional attraction. In ordinary situations, one deals with two gradients (driving forces): gradients of mass velocity and temperature, and two responses: boundary mass and heat flows. Therefore, one has four slip coefficients that relate the responses (flows) to the driving forces (gradients). As a result, the slip coefficients form a 2×2 matrix [two off-diagonal coefficients obey the Onsager relation and are not independent]. In spin-polarized quantum systems one has an additional driving force—the magnetization gradient, and an additional response—the boundary spin current. Now the slip coefficients form a 3×3 matrix with 3 independent off-diagonal coefficients, resulting in more diverse phenomena.

Another attractive feature of spin-polarized quantum systems is the possibility to pursue a model-free description. The ultra-quantum nature of low-temperature helium and hydrogen systems allows one to develop a fairly accurate theory despite all the complexity of the underlying processes. What is more, the spin polarization makes the macroscopic manifestations of quantum phenomena more noticeable and accessible than in more traditional quantum systems. On the other hand, a consistent experimental verification of the theoretical analysis becomes more straightforward since

some of the most accurate experimental approaches are based on the NMR technique.

The exact values of slip coefficients are seldom known. The most detailed information was obtained for rarefied gases¹²⁻¹⁸ with possible applications to vacuum technology, high altitude flights and space research. We know of very few experimental and theoretical attempts (see Refs. 9, 11, and 19-26 and references therein) to describe slip phenomena in quantum liquids and gases.

In this paper we study boundary slip in spin-polarized Fermi liquids and quantum gases. The spin polarization makes the systems analogous to two-component mixtures (of up and down spins). As a result, the number of equations doubles. For Fermi liquids the calculations are based on a gas-like character of the quasi-particles. Additional simplifications are related to large wavelengths of particles of quantum gases.

We want to determine the slip length, derive proper Onsager relations, calculate the values of slip coefficients, and to apply the results to longitudinal spin diffusion in spin-polarized quantum systems such as normal liquid ${}^3\text{He}\uparrow$, liquid and gaseous ${}^3\text{He}\uparrow$ - ${}^4\text{He}$ mixtures, ${}^3\text{He}\uparrow$ gas, and $\text{H}\downarrow$ and $\text{D}\downarrow$ systems, etc. As a by-product, we will get a theory of slip phenomena in binary mixtures of quantum or classical gases for which the concentration of the mixture plays the role of the spin polarization, while the spin diffusion current should be substituted by the usual diffusion.

We start from the general spin diffusion equations of Ref. 11 with viscous renormalizations of the type²⁷ for spin pressure diffusion terms (see also the review⁹). This allows us to get proper Onsager relations and to evaluate the influence of slip terms on hydrodynamic and diffusion flow in spin-polarized systems. Our calculation of slip coefficients is based on the methods developed in Refs. 22 and 24. We use the transport equation in relaxation time approximation with diffuse boundary conditions. The accuracy of the relaxation time approximation for spin-polarized quantum gases and liquids is reasonably high (see, e.g., review⁹). The applicability and the limitations of the diffuse boundary conditions are less clear.^{20, 26} However, the diffuse boundary condition is still one of the simplest universal boundary conditions which combines effects of surface roughness and energy and momentum accommodation. There are certain indications that these effects may have somewhat different influence on boundary slip.^{19, 20, 26} We plan to return to this problem later.

2. MAIN DEFINITIONS

The presence of a new variable, namely, the spin polarization, leads to the dependence of already known slip coefficients on spin polarization or

magnetic field, and to the increase in number of equations with several new coefficients.

We are interested in “longitudinal” effects which are not accompanied by changes in the direction of magnetization. For such processes, a spin-polarized Fermi liquid (or a gas) is similar to a binary mixture of spin-up and spin-down components.⁹ Then most of the effects can be described using the terminology inherent to binary mixtures. The degree of spin polarization α plays a role similar to the concentration of the mixture of the (spin) components c which are denoted by the indices (\pm):

$$c = \frac{m_+ N_+}{m_+ N_+ + m_- N_-}, \quad \alpha = \frac{N_+ - N_-}{N_+ + N_-} = \frac{c/m_+ - (1-c)/m_-}{c/m_+ + (1-c)/m_-} \quad (1)$$

It is convenient to use as an independent variable not the concentration c or polarization α , but the chemical potential μ which is conjugate to c and is expressed via the chemical potentials of (spin) components μ_{\pm} as

$$\nabla\mu = \frac{1}{m_+} \nabla\mu_+ - \frac{1}{m_-} \nabla\mu_- \quad (2)$$

We are interested mostly in spin-polarized quantum gases for which, with a rather high accuracy, $m_+ = m_-$.⁹ However, our results are applicable for any two-component gases for which masses of particles belonging to different components m_+ and m_- may be different. The same is true for spin-polarized Fermi liquids. For this reason, we will keep two different masses in Eq. (2).

The possible difference in masses of particles from different spin components forces us to make the following choice. We can work either with mass concentration c and mass currents, or with particle concentration (like α) and particle flows. The former approach is standard for hydrodynamics of mixtures, while the latter seems to be more suitable for spin dynamics. Though the results for both approaches are similar, the transition from one representation to another can be rather annoying, especially if $m_+ \neq m_-$. Throughout this paper, in contrast to Ref. 11, we will follow consistently the hydrodynamic approach and work with mass concentration and mass (diffusion) currents exclusively.

Bulk mass diffusion current is defined through the mass currents of the spin components as

$$\mathbf{j}_D \equiv \mathbf{j}_+ - \mathbf{j}_- = -D\rho \nabla c, \quad (3)$$

and can be expressed via $\nabla\mu$ using the relations

$$\nabla N_{\pm} = -\nabla\mu_{\pm} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\partial f_{\pm}^{(0)}}{\partial \varepsilon} \quad (4)$$

where $f_{\pm}^{(0)}$ are the equilibrium distribution functions for different (spin) components. Note, that the spin diffusion current \mathbf{j}_s and the spin diffusion coefficient D_s are usually defined in a form which is somewhat different from Eq. (3):

$$\mathbf{j}_s \equiv \mathbf{j}_+/m_+ - \mathbf{j}_-/m_- = -D_s N \nabla \alpha$$

Slip boundary conditions express boundary values of hydrodynamic flows through the gradients of hydrodynamic variables. In case of spin-polarized systems, the general slip boundary conditions should have the following form (cf. Refs. 11 and 28):

$$\begin{aligned} \mathbf{u}_t(z=0) &= \alpha_{11} \frac{\partial \mathbf{u}_t(z \rightarrow 0)}{\partial z} + \alpha_{12} \nabla_t \mu(z=0) + \alpha_{13} \nabla_t T(z=0) \\ \mathbf{J}_t^{(s)}(z=0) &= \alpha_{21} \frac{\partial \mathbf{u}_t(z \rightarrow 0)}{\partial z} + \alpha_{22} \nabla_t \mu(z=0) + \alpha_{23} \nabla_t T(z=0) \quad (5) \\ \mathbf{Q}_t^{(s)}(z=0) &= \alpha_{31} \frac{\partial \mathbf{u}_t(z \rightarrow 0)}{\partial z} + \alpha_{32} \nabla_t \mu(z=0) + \alpha_{33} \nabla_t T(z=0) \end{aligned}$$

where the index t marks the components of vectors along the boundary $z=0$, \mathbf{u} is the hydrodynamic (mass) velocity, $\mathbf{J}_t^{(s)}$ and $\mathbf{Q}_t^{(s)}$ are the additional surface mass diffusion current of spin components and heat flow caused by the boundary slip, α_{ik} is the matrix of slip coefficients. The off-diagonal components of the matrix α_{ik} are related to each other via Onsager relations (see Appendix A). The currents $\mathbf{J}^{(s)}$ and $\mathbf{Q}^{(s)}$ are defined as

$$\mathbf{J}^{(s)} = \int_0^{\infty} dz [\mathbf{j}_D(z) - \mathbf{j}_D(\infty)], \quad \mathbf{Q}^{(s)} = \int_0^{\infty} dz [\mathbf{q}(z) - \mathbf{q}(\infty)] \quad (6)$$

where \mathbf{q} is the density of the bulk heat current. In what follows, we will determine the values of the slip coefficients α_{ik} . In linear systems we can calculate all slip coefficients one by one, independently of each other.

3. SLIP LENGTH AND RELATED COEFFICIENTS

Let us start from the most commonly used coefficient, namely the slip length $\xi = \alpha_{11}$. The slip length describes the difference between the real boundary value of the mass velocity $\mathbf{u}(z=0)$ and the hydrodynamic boundary condition $u(0)=0$ (see Fig. 1). The diffuse scattering from the wall $z=0$ affects the velocity profile $\mathbf{u}(z) = \hat{\mathbf{x}}u(z)$. The deviation $g_z(\mathbf{p})$ of the

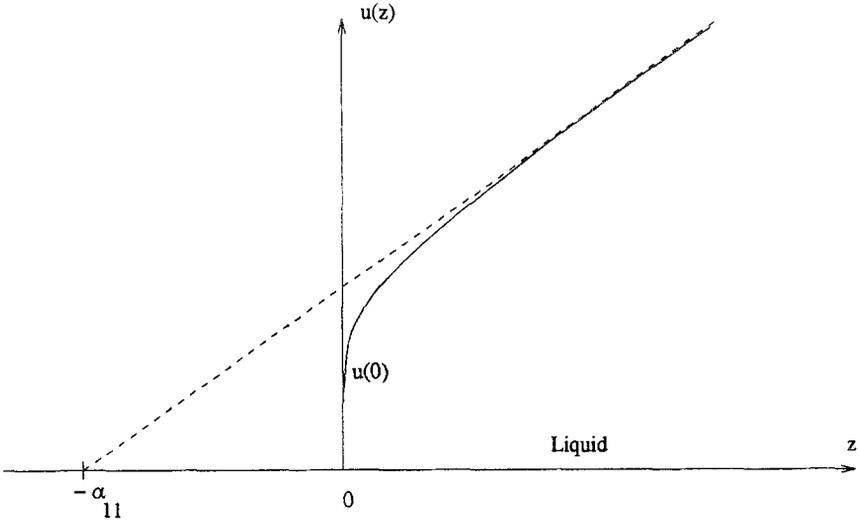


Fig. 1. The schematic velocity profile $u(z)$ near the wall $z=0$. The slip length $\xi = a_{11}$ is determined by the intersection of the tangent to the curve with the axis z .

distribution function $f_\alpha(\mathbf{p})$ of the type α (spin-ups or spin-downs) from the local equilibrium $f_\alpha^{(0)}(\varepsilon - \mathbf{p}\mathbf{u})$ satisfies the transport equation

$$v_z \frac{\partial g_\alpha}{\partial z} - v_z p_x u'(z) \frac{\partial f_\alpha^{(0)}}{\partial \varepsilon} = -\tau_{\alpha\beta}^{-1} g_\beta \quad (7)$$

In many cases, including dilute spin-polarized quantum gases and Fermi liquids, the matrix of inverse relaxation times $\tau_{\alpha\beta}^{-1}$ is diagonal with known diagonal elements $1/\tau_+$ and $1/\tau_-$ (see Refs. 2 and 9). However, to keep the results applicable for arbitrary two-component mixtures, we will work with an arbitrary symmetric matrix $\tau_{\alpha\beta}^{-1}$. The characteristic numbers for Eq. (7) are

$$\lambda_{1,2} = -\frac{1}{2} \left(\frac{1}{\tau_+} + \frac{1}{\tau_-} \right) \pm \frac{1}{2} \left[\left(\frac{1}{\tau_+} - \frac{1}{\tau_-} \right)^2 + \frac{4}{\tau_{+-}\tau_{-+}} \right]^{1/2}$$

and the general solution of Eq. (7) has the form

$$g_\alpha(z) = e^{-\tau_{\alpha\beta}^{-1} z/v_z} g_\beta(0) + \int_0^z dz' p_x u'(z') \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} e^{-\tau_{\alpha\beta}^{-1} (z-z')/v_z} \quad (8)$$

with the matrix

$$e^{-\tau_{\alpha\beta}^{-1} z/v_z} \equiv U_{\alpha\gamma}^{-1} e^{-\lambda_\gamma z/v_z} U_{\gamma\beta} \quad (9)$$

where \hat{U} is the unitary matrix which diagonalizes $\hat{\tau}^{-1}$. The diffuse scattering by the wall corresponds to the following boundary conditions at $z = 0$:

$$g_\alpha(z=0, v_z > 0) = p_x u(0) \frac{\partial f_\alpha^{(0)}}{\partial \varepsilon}$$

$$g_\alpha(z=0, v_z < 0) = -p_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_0^\infty dz' u'(z') e^{\tau_{\alpha\beta}^{-1} z'/v_z}$$

(the second equation is equivalent to the condition $g(v_z < 0, z \rightarrow \infty) = 0$). The solution (8) of the transport equation (7) with these boundary conditions is

$$g_\alpha(z, v_z > 0) = p_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \left[u(0) e^{-\tau_{\alpha\beta}^{-1} z/v_z} + \int_0^z dz' u'(z') e^{-\tau_{\alpha\beta}^{-1} (z-z')/v_z} \right]$$

$$g_\alpha(z, v_z < 0) = -p_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_z^\infty dz' u'(z') e^{\tau_{\alpha\beta}^{-1} (z'-z)/v_z}$$
(10)

Following Refs. 22 and 24, we parametrize the derivative of the mass velocity $u(z)$ as

$$u'(z) = u'_\infty [1 + \psi(z)] \quad (11)$$

Then, by definition (see Fig. 1), the slip length $\xi = \alpha_{11}$ is equal to

$$\alpha_{11} = \frac{u(0)}{u'_\infty} + \int_0^\infty dz \psi(z) \quad (12)$$

where $u'_\infty = u'(z \rightarrow \infty)$, and the function $\psi(z)$ is positive and small.

Since all the currents are time-independent, the total momentum flow of both components satisfies the following equation:

$$\frac{\partial \Pi_{xi}}{\partial r_i} \equiv \frac{\partial \Pi_{xz}(z)}{\partial z} = 0 \quad (13)$$

and is constant across the liquid. At $z \rightarrow \infty$ this flow is equal to $-\eta u'_\infty$, and therefore, can be written at arbitrary z as

$$\Pi_{xz} = -\eta u'_\infty = \sum_\alpha \int \frac{d^3 p}{(2\pi\hbar)^3} v_z p_x g_\alpha(\mathbf{p}, z) \quad (14)$$

where $\eta = \eta_+ + \eta_-$ is the viscosity [definitions of partial viscosities of spin components η_α and their values for two-component quantum liquids and gases are given in Refs. 9 and 11].

With the help of the functions $L_n^{(\alpha)}(z)$,

$$L_n^{(\alpha)}(z) = - \int_{v_z > 0} \frac{d^3 p}{(2\pi\hbar)^3} \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} p_x^2 v_z^n \tau_{\alpha\beta}^{n-1} e^{-\tau_\beta^{-1} z/v_z} \quad (15)$$

Eq. (14) can be written as

$$\begin{aligned} -\eta u'_\infty = \sum_\alpha \left\{ -u(0) L_1^{(\alpha)}(z) - 2u'_\infty L_2^{(\alpha)}(0) + u'_\infty L_2^{(\alpha)}(z) \right. \\ \left. - u'_\infty \int_0^\infty dz' \psi(z') L_1^{(\alpha)}(|z-z'|) \right\} \quad (16) \end{aligned}$$

On the other hand, at $z \rightarrow \infty$ the derivative $\partial g/\partial z = 0$, and, according to Eq. (7),

$$g_\alpha(\infty) = u'_\infty \tau_{\alpha\beta} \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} p_x v_z \quad (17)$$

leading to the following expression for partial viscosities through the functions L_n :

$$\eta_\alpha \equiv - \int \frac{d^3 p}{(2\pi\hbar)^3} p_x v_z g_\alpha(\infty) = 2L_2^{(\alpha)}(0) \quad (18)$$

Eqs. (12) and (18) should be substituted into Eq. (16):

$$\begin{aligned} \alpha_{11} \sum_\alpha L_1^{(\alpha)}(z) - \sum_\alpha L_2^{(\alpha)}(z) \\ = - \int_0^\infty dz' \psi(z') \sum_\alpha [L_1^{(\alpha)}(|z-z'|) - L_1^{(\alpha)}(z)] \quad (19) \end{aligned}$$

or, at $z = 0$,

$$\begin{aligned} \alpha_{11} \sum_\alpha L_1^{(\alpha)}(0) - \sum_\alpha L_2^{(\alpha)}(0) \\ = - \int_0^\infty dz' \psi(z') \sum_\alpha [L_1^{(\alpha)}(z') - L_1^{(\alpha)}(0)] \quad (20) \end{aligned}$$

We can use this expression to calculate a lower bound for α_{11} . Since the expression in square brackets in Eq. (20) is obviously negative, and $\psi(z) > 0$, the slip length satisfies the following inequality:

$$\alpha_{11} > \frac{\sum_\alpha L_2^{(\alpha)}(0)}{\sum_\alpha L_1^{(\alpha)}(0)} \quad (21)$$

The values of $L_n(0)$ for different temperature regimes are given in Appendix B.

To determine an upper bound for the slip length α_{11} , we should integrate Eq. (19) using the identity $dL_n^{(\alpha)}(z)/dz = -L_{n-1}^{(\alpha)}(z)$:

$$\begin{aligned} \alpha_{11} \sum_{\alpha} L_2^{(\alpha)}(0) - \sum_{\alpha} L_3^{(\alpha)}(0) \\ = \int_0^{\infty} dz' \psi(z') \sum_{\alpha} [L_2^{(\alpha)}(z') - L_2^{(\alpha)}(0)] \end{aligned} \quad (22)$$

This immediately gives us

$$\alpha_{11} < \frac{\sum_{\alpha} L_3^{(\alpha)}(0)}{\sum_{\alpha} L_2^{(\alpha)}(0)} \quad (23)$$

The functions $L_n(0)$ are given in Appendix B. In the same way as in Ref. 22, we can find a somewhat improved lower bound:

$$\alpha_{11} > \frac{1}{2} \frac{\sum_{\alpha} L_2^{(\alpha)}(0)}{\sum_{\alpha} L_1^{(\alpha)}(0)} + \frac{1}{2} \frac{\sum_{\alpha} L_3^{(\alpha)}(0)}{\sum_{\alpha} L_2^{(\alpha)}(0)} \quad (24)$$

Let us turn now to the off-diagonal slip coefficient α_{21} (5). The bulk mass diffusion current for spin components (3) has only the x -component and can be defined as

$$j_D = j_+ - j_-, \quad j_{\alpha} = \int \frac{d^3 p}{(2\pi\hbar)^3} p_x g_{\alpha}(\mathbf{p}, z) \quad (25)$$

with functions g_{α} given by Eq. (8). Obviously, there is no diffusion current at $z \rightarrow \infty$, and all spin diffusion (25) is caused by the surface slip exclusively. The total additional surface diffusion current $J^{(s)}$ (5), (6) gives the slip coefficient $\alpha_{21} = J^{(s)}/u'_{\infty}$.

After some simple transformations with the help of Eqs. (10) and (11), the expression for diffusion current reduces to

$$\begin{aligned} \frac{j_{\alpha}(z)}{u'_{\infty}} = -\frac{u(0)}{u'_{\infty}} K_1^{(\alpha)}(z) + K_2^{(\alpha)}(z) \\ - \int_0^z dz' \psi(z') K_1^{(\alpha)}(z-z') + \int_z^{\infty} dz' \psi(z') K_1^{(\alpha)}(|z'-z|) \end{aligned} \quad (26)$$

The functions K_n in Eq. (26) are very similar to the functions L_n above, Eq. (15):

$$K_n^{(\alpha)}(z) = - \int_{v_z > 0} \frac{d^3 p}{(2\pi\hbar)^3} \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} p_x^2 v_z^{n-1} \tau_{xy}^{n-1} e^{-\tau_{xy}^{-1} z/v_z} \quad (27)$$

After integration in Eq. (6) we get

$$\begin{aligned} \frac{J_\alpha^{(s)}(z)}{u'_\infty} &= \frac{u(0)}{u'_\infty} [K_2^{(\alpha)}(z) - K_2^{(\alpha)}(0)] - [K_3^{(\alpha)}(z) - K_3^{(\alpha)}(0)] \\ &\quad - \int_0^z dz'' \int_0^{z''} dz' \psi(z') K_1^{(\alpha)}(z'' - z') \\ &\quad + \int_0^z dz'' \int_{z''}^\infty dz' \psi(z') K_1^{(\alpha)}(|z' - z''|) \end{aligned} \quad (28)$$

In the last two integrals we can change the order of integration and perform a simple integration over z'' :

$$\begin{aligned} \frac{J_\alpha^{(s)}(z)}{u'_\infty} &= \frac{u(0)}{u'_\infty} [K_2^{(\alpha)}(z) - K_2^{(\alpha)}(0)] - [K_3^{(\alpha)}(z) - K_3^{(\alpha)}(0)] \\ &\quad - \int_0^z dz' \psi(z') K_2^{(\alpha)}(0) + \int_0^z dz' \psi(z') K_2^{(\alpha)}(z' - z) \\ &\quad + \int_z^\infty dz' \psi(z') K_2^{(\alpha)}(|z' - z|) + \int_0^z dz' \psi(z') K_2^{(\alpha)}(0) \\ &\quad - \int_0^\infty dz' \psi(z') K_2^{(\alpha)}(z') \end{aligned} \quad (29)$$

After some algebra, the last equation is transformed to

$$\begin{aligned} \frac{J_\alpha^{(s)}(z)}{u'_\infty} &= \frac{u(0)}{u'_\infty} [K_2^{(\alpha)}(z) - K_2^{(\alpha)}(0)] - [K_3^{(\alpha)}(z) - K_3^{(\alpha)}(0)] \\ &\quad + \int_0^\infty dz' \psi(z') [K_2^{(\alpha)}(|z - z'|) - K_2^{(\alpha)}(z')] \end{aligned} \quad (30)$$

and at $z \rightarrow \infty$ reduces to the following form

$$\begin{aligned} \alpha_{21} &\equiv \frac{J_+^{(s)}(\infty) - J_-^{(s)}(\infty)}{u'_\infty} = - \frac{u(0)}{u'_\infty} [K_2^+(0) - K_2^-(0)] \\ &\quad + K_3^+(0) - K_3^-(0) - \int_0^\infty dz' \psi(z') [K_2^+(z') - K_2^-(z')] \end{aligned} \quad (31)$$

Now we can substitute $u(0)$ by α_{11} using Eq. (12):

$$\begin{aligned} \alpha_{21} = & -\alpha_{11}[K_2^+(0) - K_2^-(0)] + K_3^+(0) - K_3^-(0) \\ & - \int_0^\infty dz' \psi(z') [K_2^+(z') - K_2^+(0) - K_2^-(z') + K_2^-(0)] \end{aligned} \quad (32)$$

With the help of Eq. (22) this equation can be transformed into

$$\begin{aligned} \alpha_{21} = & -(K_2^+(0) - K_2^-(0)) \left[\frac{L_2^+(0) + L_2^-(0)}{L_1^+(0) + L_1^-(0)} - \frac{K_3^+(0) - K_3^-(0)}{K_2^+(0) - K_2^-(0)} \right. \\ & \left. - \int dz \psi(z) \left(\frac{L_1^+(z) + L_1^-(z)}{L_1^+(0) + L_1^-(0)} - \frac{K_2^+(z) - K_2^-(z)}{K_2^+(0) - K_2^-(0)} \right) \right] \end{aligned}$$

It can easily be checked that the integrand in this equation is positive. This gives us an obvious upper bound for α_{21} :

$$\alpha_{21} < -(K_2^+(0) - K_2^-(0)) \left[\frac{L_2^+(0) + L_2^-(0)}{L_1^+(0) + L_1^-(0)} - \frac{K_3^+(0) - K_3^-(0)}{K_2^+(0) - K_2^-(0)} \right] \quad (33)$$

We can get another bound by writing Eq. (26)

$$0 > \frac{j_D(0)}{u'_\infty} > -\alpha_{11}[K_1^+(0) - K_1^-(0)] + K_2^+(0) - K_2^-(0) \quad (34)$$

On the other hand (see Fig. 2),

$$J^{(s)} \equiv \int_0^\infty j_D(z) dz < \frac{1}{2} j_D(0) \zeta < 0 \quad (35)$$

where the length ζ is given by

$$\zeta = -\frac{j_D(0)}{j'_D(0)} \quad (36)$$

According to Eq. (26) (cf. Eq. (34)),

$$\frac{j'_D(0)}{u'_\infty} < \alpha_{11}[K_0^+(0) - K_0^-(0)] - [K_1^+(0) - K_1^-(0)]$$

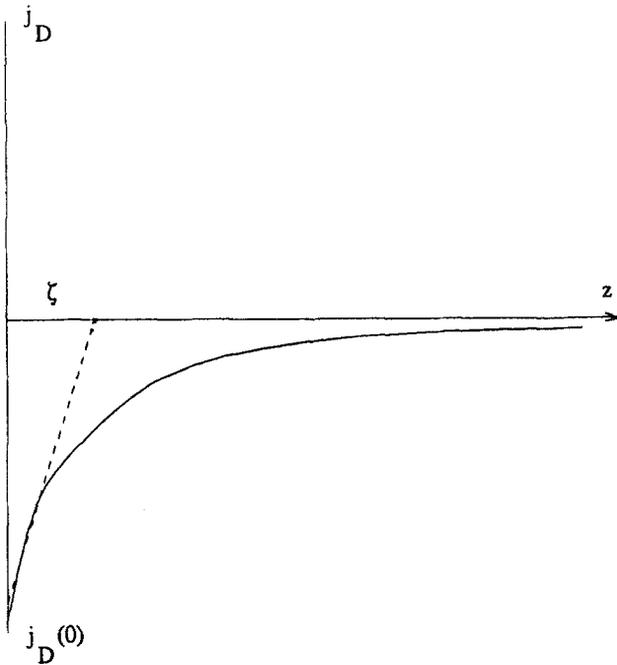


Fig. 2. The schematic profile of the diffusion current $j_D(z)$ near the wall $z=0$. The current density at $z \rightarrow \infty$ is zero. The total slip diffusion current $J^{(s)}$ is the area between the curve and the axis z . The length ζ (36) is determined by the intersection of the tangent to the curve and the axis z .

This condition, with the help of Eq. (35), gives another upper bound for α_{21} :

$$\begin{aligned} \alpha_{21} &< -\frac{1}{2} \frac{j_D^2(0)}{u'_\infty j'_D(0)} \\ &< -\frac{1}{2} \frac{[\alpha_{11}(K_1^+(0) - K_1^-(0)) - (K_2^+(0) - K_2^-(0))]^2}{\alpha_{11}[K_0^{+2}(0) - K_0^{-2}(0)] - (K_1^+(0) - K_1^-(0))} \end{aligned} \quad (37)$$

Other bounds for the coefficient α_{21} can be obtained by calculating the bounds for the coefficient α_{12} and using the Onsager relation between the coefficients α_{21} and α_{12} (see Appendix A, Eq. (A.16)):

$$\alpha_{21} = \eta \alpha_{12} + \left(\frac{\partial \mu}{\partial c} \right)^{-1} \frac{\rho^2 D \eta}{2 \rho_+ \rho_-} \left[\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right], \quad \rho_\pm \equiv m_\pm N_\pm \quad (38)$$

The lower bound for α_{12} and, therefore, for α_{21} (38), will be given in the next Section.

Now let us turn to the coefficient α_{31} in the boundary conditions (5). The partial heat currents of the spin components are defined as

$$q_\alpha = \int \frac{d^3p}{(2\pi\hbar)^3} v_x \varepsilon(\mathbf{p}) g_\alpha(\mathbf{p}, z), \quad q = q_+ + q_- \quad (39)$$

where g_\pm are again the solutions of the transport equation (10). We will define α_{31} through the surface heat flow $Q^{(s)}$ (6), $\alpha_{31} = Q^{(s)}/u'_\infty$. The calculation of the lower bound for α_{31} is exactly the same as for α_{21} . Then the only differences between the bound (33) for α_{21} and the corresponding equation for α_{31} are the sign between the contributions spin-ups and spin-downs and the use of integrals M^\pm ,

$$M_n^{(\alpha)}(z) = - \int_{v_z > 0} \frac{d^3p}{(2\pi\hbar)^3} \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} v^2 p_x^2 v_z^{n-1} \tau_{\alpha\gamma}^{n-1} e^{-\tau_{\gamma\beta}^{-1} z/v_z} \quad (40)$$

instead of K^\pm (27):

$$2\alpha_{31} > -\alpha_{11} [M_2^+(0) + M_2^-(0)] + M_3^+(0) + M_3^-(0) \quad (41)$$

We can easily get a bound which is similar to Eq. (37):

$$\begin{aligned} 2\alpha_{31} &< -\frac{1}{2} \frac{q^2(0)}{u'_\infty q'(0)} \\ &< -\frac{1}{2} \frac{[\alpha_{11}(M_1^+(0) + M_1^-(0)) - (M_2^+(0) + M_2^-(0))]^2}{\alpha_{11}[M_0^{+2}(0) + M_0^{-2}(0)] - (M_1^+(0) + M_1^-(0))} \end{aligned} \quad (42)$$

Later we will determine another upper bound for α_{31} by calculating the upper bound for α_{13} and using the Onsager relation from Appendix A. This will be done in Sec. 5.

4. SPIN DIFFUSION AND SURFACE SLIP

The next step is to calculate slip coefficients related to spin diffusion, α_{22} , α_{12} , and α_{32} , Eq. (5). Here we assume that the bulk gradient of spin polarization is associated only with $\nabla_x \mu(z)$, Eqs. (2) and (3). Then the transport equation assumes the form

$$v_z \frac{\partial g_\alpha}{\partial z} - v_x \nabla_x \mu_\alpha \frac{\partial f_\alpha^{(0)}}{\partial \varepsilon} = -\tau_{\alpha\beta}^{-1} g_\beta \quad (43)$$

with the diffuse boundary conditions:

$$g_\alpha(z=0, v_z > 0) = 0, \quad (44)$$

$$g_\alpha(z=0, v_z < 0) = -\frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_0^\infty dz' \nabla_x \mu_\alpha \frac{v_x}{v_z} e^{\tau_{\alpha\beta}^{-1} z'/v_z}$$

The solution of Eqs. (43) with the boundary conditions (44) has the form

$$g_\alpha(z, v_z > 0) = v_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_0^z \frac{dz'}{v_z} \nabla_x \mu_\beta e^{-\tau_{\alpha\beta}^{-1} (z-z')/v_z} \quad (45)$$

$$g_\alpha(z, v_z < 0) = -v_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_z^\infty \frac{dz'}{v_z} \nabla_x \mu_\beta e^{\tau_{\alpha\beta}^{-1} (z'-z)/v_z}$$

The mass and heat currents for different spin components, Eqs. (25) and (39), will be expressed via integrals

$$\left\{ \begin{array}{l} Q_n^\alpha(z) \\ V_n^\alpha(z) \end{array} \right\} = - \int_{v_z > 0} \frac{d^3 p}{(2\pi\hbar)^3} \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} p_x v_x v_z^{n-2} \tau_{\alpha\beta}^{n-1} e^{-\tau_{\alpha\beta}^{-1} z/v_z} \left\{ \begin{array}{l} 1 \\ v^2 \end{array} \right\} \quad (46)$$

With the notations

$$\nabla_x \mu_\alpha(z) = \nabla_x \mu_\alpha(\infty)(1 + \phi_\alpha(z)) \quad (47)$$

the currents (25), (39) can be written as

$$\frac{j_\alpha(z)}{\nabla_x \mu_\alpha(\infty)} = -2Q_2^\alpha(0) + Q_2^\alpha(z) - \int_0^\infty dz' \phi_\alpha(z') Q_1^\alpha(|z-z'|) \quad (48)$$

$$2 \frac{q_\alpha(z)}{\nabla_x \mu_\alpha(\infty)} = -2V_2^\alpha(0) + V_2^\alpha(z) - \int_0^\infty dz' \phi_\alpha(z') V_1^\alpha(|z-z'|)$$

where ϕ is negative, $\phi(z) < 0$, and small.

Let us now use the condition that the total bulk mass current $\mathbf{j} = \mathbf{j}_+ + \mathbf{j}_-$ goes to zero at $z \rightarrow \infty$, while the total mass diffusion current of spin components, $\mathbf{j}_D = \mathbf{j}_+ - \mathbf{j}_-$ is non-zero and equal to $-\rho D \nabla_x c$:

$$Q_2^+(0) \nabla_x \mu_+(\infty) + Q_2^-(0) \nabla_x \mu_-(\infty) = 0$$

$$-2(Q_2^+(0) \nabla_x \mu_+(\infty) - Q_2^-(0) \nabla_x \mu_-(\infty)) \quad (49)$$

$$= -\rho D \left(\frac{\partial \mu}{\partial c} \right)_{P,T}^{-1} \nabla_x \mu(\infty)$$

where $\nabla_x \mu \equiv \nabla \mu_+ / m_+ - \nabla \mu_- / m_-$, Eq. (2). According to Eq. (49),

$$\frac{\nabla_x \mu_{\pm}(\infty)}{m_{\pm}} = \pm \frac{m_{\mp} Q_2^{\mp}(0)}{m_+ Q_2^+(0) + m_- Q_2^-(0)} \nabla_x \mu(\infty) \quad (50)$$

The bulk spin diffusion coefficient D can be expressed via the functions Q_n as

$$\rho D \left(\frac{\partial \mu}{\partial c} \right)_{P,T}^{-1} = 4m_+ m_- \frac{Q_2^+(0) Q_2^-(0)}{m_+ Q_2^+(0) + m_- Q_2^-(0)} \quad (51)$$

Now we can estimate the coefficient α_{22} . The part of the total spin diffusion current, which is related to the surface slip, is equal to

$$\begin{aligned} & 4 \frac{j_D(z) - j_D(\infty)}{\rho D (\partial c / \partial \mu) \nabla_x \mu(\infty)} \\ &= \frac{Q_2^+(z)}{Q_2^+(0)} + \frac{Q_2^-(z)}{Q_2^-(0)} - \int_0^{\infty} dz' \frac{Q_1^+(|z-z'|)}{Q_2^+(0)} \phi^+(z') \\ & \quad - \int_0^{\infty} dz' \frac{Q_1^-(|z-z'|)}{Q_2^-(0)} \phi^-(z') \end{aligned} \quad (52)$$

By definition,

$$\alpha_{22} \nabla_x \mu(\infty) = J^{(s)} \equiv \int_0^{\infty} (j_D(z) - j_D(\infty)) dz \quad (53)$$

and, according to Eq. (52),

$$\begin{aligned} \frac{4\alpha_{22}}{\rho D (\partial c / \partial \mu)} &= \frac{Q_3^+(0)}{Q_2^+(0)} + \frac{Q_3^-(0)}{Q_2^-(0)} + \int_0^{\infty} dz' \left[\frac{Q_2^+(z')}{Q_2^+(0)} - 2 \right] \phi^+(z') \\ & \quad + \int_0^{\infty} dz' \left[\frac{Q_2^-(z')}{Q_2^-(0)} - 2 \right] \phi^-(z') \end{aligned} \quad (54)$$

This immediately gives us the lower bound

$$\frac{4\alpha_{22}}{\rho D (\partial c / \partial \mu)} > \frac{Q_3^+(0)}{Q_2^+(0)} + \frac{Q_3^-(0)}{Q_2^-(0)} \quad (55)$$

On the other hand, integral (53) can be estimated in the same way as in Eqs. (35) and (37):

$$\begin{aligned} \alpha_{22} &> -\frac{1}{2} \frac{(j_D(0) - j_D(\infty))^2}{\nabla_x \mu(\infty) j_D'(0)} \\ &> \frac{1}{2} \rho D \frac{\partial c}{\partial \mu} \frac{Q_2^+(0) Q_2^-(0)}{Q_1^+(0) Q_2^-(0) + Q_1^-(0) Q_2^+(0)} \end{aligned} \quad (56)$$

Now let us estimate α_{12} . According to Eq. (48), the full current is

$$\begin{aligned} \frac{4(j_+(z) + j_-(z))}{\rho D(\partial c/\partial \mu) \nabla_x \mu(\infty)} &= \frac{Q_2^+(z)}{Q_2^+(0)} - \frac{Q_2^-(z)}{Q_2^-(0)} - \int_0^\infty dz' \frac{Q_1^+(|z-z'|)}{Q_2^+(0)} \phi^+(z') \\ &+ \int_0^\infty dz' \frac{Q_1^-(|z-z'|)}{Q_2^-(0)} \phi^-(z') \end{aligned} \quad (57)$$

By definition,

$$\alpha_{12} \nabla_x \mu(\infty) = [j_+(0) + j_-(0)](m_+ N_+ + m_- N_-)^{-1} \quad (58)$$

Numerical analysis of Eq. (57) at $z=0$ shows that

$$\alpha_{12} > 0 \quad (59)$$

This is equivalent to the following bound for α_{21} (38):

$$\alpha_{21} > -\left(\frac{\partial \mu}{\partial c}\right)^{-1} \frac{\rho^2 D \eta}{2\rho_+ \rho_-} \left[\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right] \quad (60)$$

The additional bounds for α_{12} are given by the bounds (33), (37) for α_{21} and the Onsager relation (38).

The total heat current (39), (48) is very similar to Eq. (52) and is equal to

$$\begin{aligned} \frac{8(q_+(z) + q_-(z))}{\rho D(\partial c/\partial \mu) \nabla_x \mu(\infty)} &= \frac{V_2^+(z)}{Q_2^+(0)} - \frac{V_2^-(z)}{Q_2^-(0)} - \int_0^\infty dz' \frac{V_1^+(|z-z'|)}{Q_2^+(0)} \phi^+(z') \\ &+ \int_0^\infty dz' \frac{V_1^-(|z-z'|)}{Q_2^-(0)} \phi^-(z'). \end{aligned} \quad (61)$$

Therefore, the coefficient α_{32} which is defined similarly to Eq. (53), has the upper bound which is similar to Eq. (55):

$$\alpha_{32} < \frac{1}{4} \rho D(\partial c/\partial \mu)_{P, T} \left[\frac{V_3^+(0)}{Q_2^+(0)} - \frac{V_3^-(0)}{Q_2^-(0)} \right] \quad (62)$$

The lower bound can be obtained from the lower bound for α_{23} using the Onsager relation $T\alpha_{23} = \alpha_{32}$. Another bound is similar to Eq. (56):

$$\begin{aligned} \alpha_{32} &> -\frac{1}{2} \frac{(q(0) - q(\infty))^2}{\nabla_x \mu(\infty) q'(0)} \\ &> \frac{1}{2} \frac{\rho D Q_2^+(0) Q_2^-(0)}{V_1^+(0) Q_2^-(0) - V_1^-(0) Q_2^+(0)} \left(\frac{\partial c}{\partial \mu} \right)_{P,T} \end{aligned}$$

5. THERMAL DIFFUSION AND SURFACE SLIP

In this Section we will calculate slip coefficients related to thermal diffusion, α_{33} , α_{13} , and α_{23} , Eq. (5). We do not consider the temperature jump (an analog of Kapitza resistance in the kinetic theory of gases), and assume that the sole bulk gradient is $\nabla_x T(z)$. The transport equation has the form

$$v_z \frac{\partial g_\alpha}{\partial z} - v_x (\varepsilon_\alpha - \mu_\alpha - T s_\alpha) \nabla_x (\ln T) \frac{\partial f_\alpha^{(0)}}{\partial \varepsilon} = -\tau_{\alpha\beta}^{-1} g_\beta \quad (63)$$

with the boundary conditions:

$$\begin{aligned} g_\alpha(z=0, v_z > 0) &= 0 \\ g_\alpha(z=0, v_z < 0) &= -\frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_0^\infty dz' \nabla_x (\ln T) (\varepsilon_\beta - \mu_\beta - T s_\beta) \frac{v_x}{v_z} e^{\tau_{\alpha\beta}^{-1} z'/v_z} \end{aligned} \quad (64)$$

where s_\pm is the entropy per particle with spin $\pm 1/2$. The solution of Eqs. (63) and (64) has the form

$$\begin{aligned} g_\alpha(z, v_z > 0) &= v_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_0^z \frac{dz'}{v_z} (\varepsilon_\beta - \mu_\beta - T s_\beta) \nabla_x (\ln T) e^{-\tau_{\alpha\beta}^{-1} (z-z')/v_z} \\ g_\alpha(z, v_z < 0) &= -v_x \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \int_z^\infty \frac{dz'}{v_z} (\varepsilon_\beta - \mu_\beta - T s_\beta) \nabla_x (\ln T) e^{\tau_{\alpha\beta}^{-1} (z'-z)/v_z} \end{aligned} \quad (65)$$

The mass and heat currents for different spin components (25), (39) contain integrals similar to (15), (27), (46)

$$\left\{ \begin{array}{l} G_n^\alpha(z) \\ S_n^\alpha(z) \end{array} \right\} = - \int_{v_z > 0} \frac{d^3 p}{(2\pi\hbar)^3} \frac{\partial f_\beta^{(0)}}{\partial \varepsilon} \varepsilon_\beta p_x v_x v_z^{n-2} \tau_{xy}^{n-1} e^{-\tau_{\alpha\beta}^{-1} z/v_z} \left\{ \begin{array}{l} 1 \\ v^2 \end{array} \right\} \quad (66)$$

The temperature gradient is parametrized as in Eq. (47):

$$\nabla_x (\ln T(z)) = \nabla_x (\ln T(\infty)) (1 + \varphi(z)) \quad (67)$$

Then the spin and heat currents (25), (39), (65) can be written as

$$\begin{aligned}
 & \frac{j_x(z)}{\nabla_x(\ln T(\infty))} \\
 &= -2G_2^\alpha(0) + G_2^\alpha(z) + (2Q_2^\alpha(0) - Q_2^\alpha(z))(\mu_\alpha + Ts_\alpha) \\
 & \quad - \int_0^\infty dz' \varphi(z') [G_1^\alpha(|z-z'|) - Q_1^\alpha(|z-z'|)(\mu_\alpha + Ts_\alpha)] \\
 & 2 \frac{q_\alpha(z)}{\nabla_x(\ln T(\infty))} \\
 &= -2S_2^\alpha(0) + S_2^\alpha(z) + (2V_2^\alpha(0) - V_2^\alpha(z))(\mu_\alpha + Ts_\alpha) \\
 & \quad - \int_0^\infty dz' \varphi_\alpha(z') [S_1^\alpha(|z-z'|) - V_1^\alpha(|z-z'|)(\mu_\alpha + Ts_\alpha)]
 \end{aligned} \tag{68}$$

where φ is negative and small. Now the total current $\mathbf{j}_+ + \mathbf{j}_-$ goes to zero when $z \rightarrow \infty$, while the bulk diffusion current $\mathbf{j}_+ - \mathbf{j}_-$ at $z \rightarrow \infty$ is finite and is equal to $-\rho k_T D \nabla \ln T$, where k_T is the (spin) thermal diffusion ratio. This means that

$$\mu_\alpha + Ts_\alpha = G_2^\alpha(0)/Q_2^\alpha(0) \mp \rho k_T D / 4Q_2^\alpha(0) \tag{69}$$

and the total heat current, $\mathbf{q} = \mathbf{q}_+ + \mathbf{q}_-$,

$$\mathbf{q}(\infty) = -\kappa \nabla T \tag{70}$$

is non-zero. The coefficient of thermal conductivity in these notations is equal to

$$\begin{aligned}
 T\kappa &= S_2^+(0) + S_2^-(0) - \frac{V_2^+(0)G_2^+(0)}{Q_2^+(0)} - \frac{V_2^-(0)G_2^-(0)}{Q_2^-(0)} \\
 & \quad + \frac{\rho k_T D}{4} \left(\frac{V_2^+(0)}{Q_2^+(0)} - \frac{V_2^-(0)}{Q_2^-(0)} \right)
 \end{aligned} \tag{71}$$

By definition,

$$\begin{aligned}
 T\rho\alpha_{13} \nabla_x(\ln T) &= j_x^+(0) + j_x^-(0), \\
 T\alpha_{23} \nabla_x(\ln T) &= \int_0^\infty dz [j_x^+(z) - j_x^-(z)] \\
 T\alpha_{33} \nabla_x(\ln T) &= \int_0^\infty dz [q_x^+(z) + q_x^-(z) - q_x^+(\infty) - q_x^-(\infty)]
 \end{aligned} \tag{72}$$

We should substitute these equations back into Eq. (68) and integrate over dz . After cumbersome transformations, similar to that in previous Sections, we get the following bounds:

$$\begin{aligned} \alpha_{13} &< 0 \\ T\alpha_{23} &< G_3^+(0) - G_3^-(0) - \frac{Q_3^+(0) G_2^+(0)}{Q_2^+(0)} + \frac{Q_3^-(0) G_2^-(0)}{Q_2^-(0)} \\ &+ \frac{\rho k_T D}{4} \left(\frac{Q_3^+(0)}{Q_2^+(0)} + \frac{Q_3^-(0)}{Q_2^-(0)} \right) \\ T\alpha_{33} &< S_3^+(0) + S_3^-(0) - \frac{V_3^+(0) G_2^+(0)}{Q_2^+(0)} - \frac{V_3^-(0) G_2^-(0)}{Q_2^-(0)} \\ &+ \frac{\rho k_T D}{4} \left(\frac{V_3^+(0)}{Q_2^+(0)} - \frac{V_3^-(0)}{Q_2^-(0)} \right) \end{aligned} \quad (73)$$

Additional bounds can also be obtained from the Onsager relations (Appendix A).

It is worth mentioning that all “thermal” slip coefficients for Fermi systems at very low temperatures contain extra powers T/T_F with respect to their “diffusion” or “viscous” counterparts, and are small. The reason is the same as for the usual bulk transport in Fermi liquids when the thermal conductivity (in dimensionless units) is small in comparison with viscosity.

6. SUMMARY

In summary, we calculated rather accurate upper and lower bounds for all nine slip coefficients (5). All the coefficients are proportional to the mean free path and, therefore, increase by the orders of magnitude at high spin polarization simultaneously with the mean free path.

The data in Appendix B provides the information on slip coefficients at arbitrary degrees of quantum degeneracy and spin polarizations. The results are especially transparent in the limiting cases of high and low temperatures, i.e. in Boltzmann and degenerate regimes. Below we will summarize the results in a somewhat simplified form.

As in Appendix B, we will assume that the matrix of relaxation times is diagonal. This is always true for dilute spin-polarized quantum gases in which the de Broglie wavelengths of particles are comparable to or larger than the radius of interaction.^{2,9} What is more, we will assume that the (effective) masses of particles are the same for both components, $m_+ = m_-$ as for ^3He in $^3\text{He}\uparrow$ - ^4He mixtures and dilute systems in general.

Even with this accuracy, the slip coefficients still remain complicated functions of polarization, $(N_+ - N_-)/N$, ratio of the relaxation times for spin-down and spin-up components, τ_-/τ_+ , and Fermi velocities, $v_{F-}/v_{F+} = (N_-/N_+)^{1/3}$. It is convenient to parametrize the results with the help of dimensionless variables

$$\nu = N_-/N_+, \quad t = \tau_-/\tau_+, \quad u = v_{F-}/v_{F+} \quad (74)$$

Unfortunately, the ratio τ_-/τ_+ is a known function of polarization only for quantum gases for which the de Broglie wavelength of particles is larger than the interaction radius, and the particle interaction reduces to the s -wave scattering. In degenerate gases under these conditions^{9, 11}

$$t \equiv \frac{\tau_-}{\tau_+} = \left[\frac{5}{2} - \frac{3}{2} \nu^{2/3} \right] \nu \frac{C(\lambda_-)}{C(\lambda_+)}, \quad \nu \equiv \frac{N_-}{N_+} \quad (75)$$

$$\lambda_- = \frac{1}{5}, \quad \lambda_+ = 1 - 2\nu^{2/3} + \frac{6}{5} \nu^{4/3}$$

where $\nu = N_-/N_+$ characterizes polarization, $(N_+ - N_-)/N = (1 - \nu)/(1 + \nu)$, $C(\lambda)$ are Brooker-Sykes correction factors (the explicit expressions are given in Ref. 11). Though these factors themselves are very important, their ratio does not change much with polarization.²⁹ Therefore, in order to simplify the results, we will neglect these corrections; according to calculations,²⁹ the loss of accuracy will not exceed 8%.

At higher temperatures, i.e. in the quantum region for spin-polarized Boltzmann gases, the ratio of relaxation times for spin-downs and spin-ups is different function of $\nu = N_-/N_+$,

$$t \equiv \frac{\tau_-}{\tau_+} = \frac{1 + 4\nu}{4 + \nu} \quad (76)$$

For better illustration, we will supplement the general results by the results in the quantum region (75), (76).

The strictest of inequalities of Sec. 3 provide the following bounds for the slip length $\xi = \alpha_{11}$ in the Boltzmann temperature range:

$$f_1 = \frac{1 + \nu t^2}{1 + \nu t} > \frac{\alpha_{11}/\tau_+}{(8T/\pi m)^{1/2}} > \frac{\pi}{8} \left[\frac{1}{\nu + 1} + \frac{\nu t}{\nu + 1} + \frac{4}{\pi} \frac{1 + \nu t^2}{1 + \nu t} \right] = f_2 \quad (77)$$

The polarization dependence of the functions $f_{1,2}$ (77) in the quantum region (76) is given in Fig. 3a.

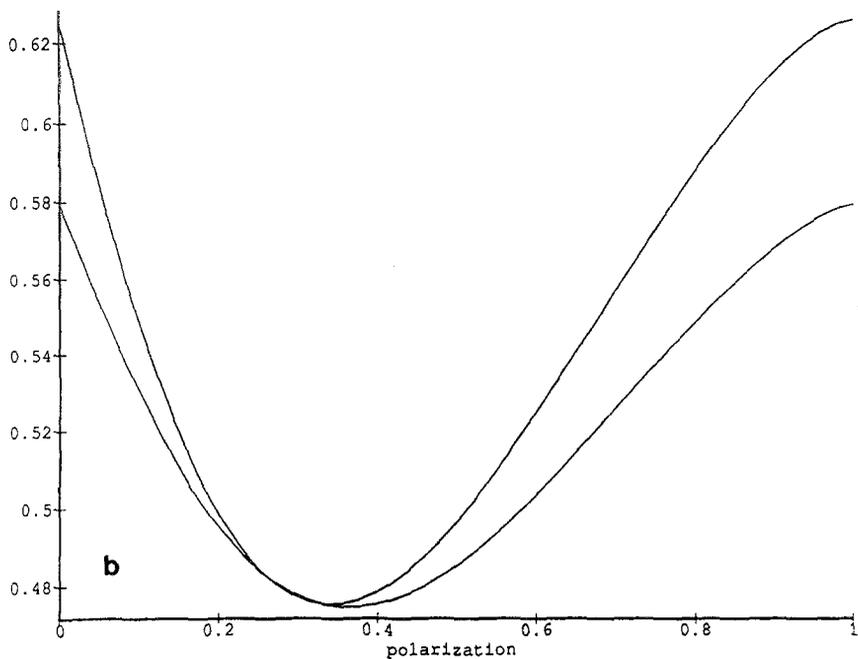
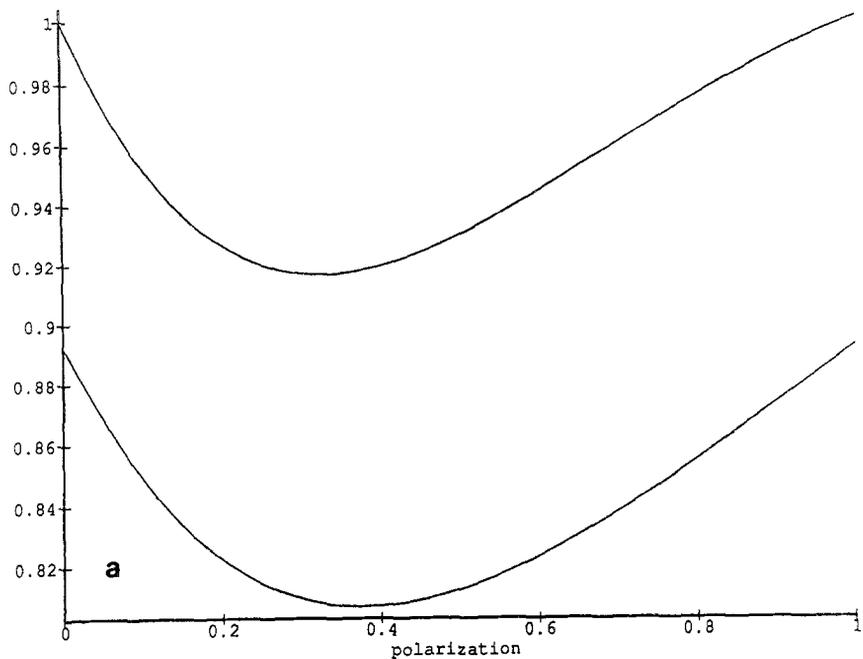


Fig. 3. Polarization dependence of the bounds for the slip length $\xi = \alpha_{11}$ for quantum gases. (a) functions $f_{1,2}$ for a quantum Boltzmann gas, Eqs. (77) and (76); (b) functions $f_{3,4}$ for degenerate gases, Eqs. (78) and (75).

At low temperatures $T \ll T_F$ the corresponding inequality obtains the form

$$\begin{aligned} f_3 &= \frac{15}{24} \frac{1 + vt^2 u^3}{1 + vt u^2} > \frac{\alpha_{11}}{\tau_+ v_{F+}} \\ &> \frac{15}{48} \frac{1 + vt^2 u^3}{1 + vt u^2} + \frac{4}{15} \frac{1 + vt u^2}{1 + vu} = f_4 \end{aligned} \quad (78)$$

The polarization dependence of the functions $f_{3,4}$ (78) in the quantum region (75) is given in Fig. 3b.

The off-diagonal slip coefficient α_{21} is negative and has the following bounds at high temperatures:

$$f_5 \equiv -\frac{vt(1-t)}{1+vt} < \frac{8\alpha_{21}}{TN_+ \tau_+^2} < \frac{v(1-t^2)}{1+v} \equiv f_6 \quad (79)$$

Functions $f_{5,6}$ in the quantum region (76) are plotted in Fig. 4a.

The corresponding bounds for the conjugate coefficient, α_{12} , are given by the Onsager relation:

$$0 < \alpha_{12} = \frac{4}{TN_+ \tau_+} \frac{\alpha_{21}}{1+vt} - \frac{\tau_+}{4} \frac{(1+v)t}{1+vt} \left(\frac{1-v}{1+v} - \frac{1-vt}{1+vt} \right)$$

In the low-temperature region,

$$\begin{aligned} f_7 &\equiv -(1+v)t \frac{1+vtu^2}{1+vt} \left(\frac{1-vtu^2}{1+vtu^2} - \frac{1-v}{1+v} \right) \\ &< \frac{5\alpha_{21}}{mN_+ \tau_+^2 v_{F+}^2} < -\frac{1}{2} \frac{1+vtu^2}{1+vu} (1-vtu) + \frac{1}{2} (1-vt^2 u^2) \\ &\equiv f_8 \end{aligned} \quad (80)$$

while the Onsager relation becomes

$$0 < \alpha_{12} = \frac{5}{mN_+ \tau_+ v_{F+}^2} \frac{\alpha_{21}}{1+vtu^2} - \frac{(1+v)\tau_+ t}{1+vt} \left(\frac{1-v}{1+v} - \frac{1-vtu^2}{1+vtu^2} \right)$$

In the s -wave scattering approximation (75) these two functions, $f_{7,8}$, are given in Fig. 4b.

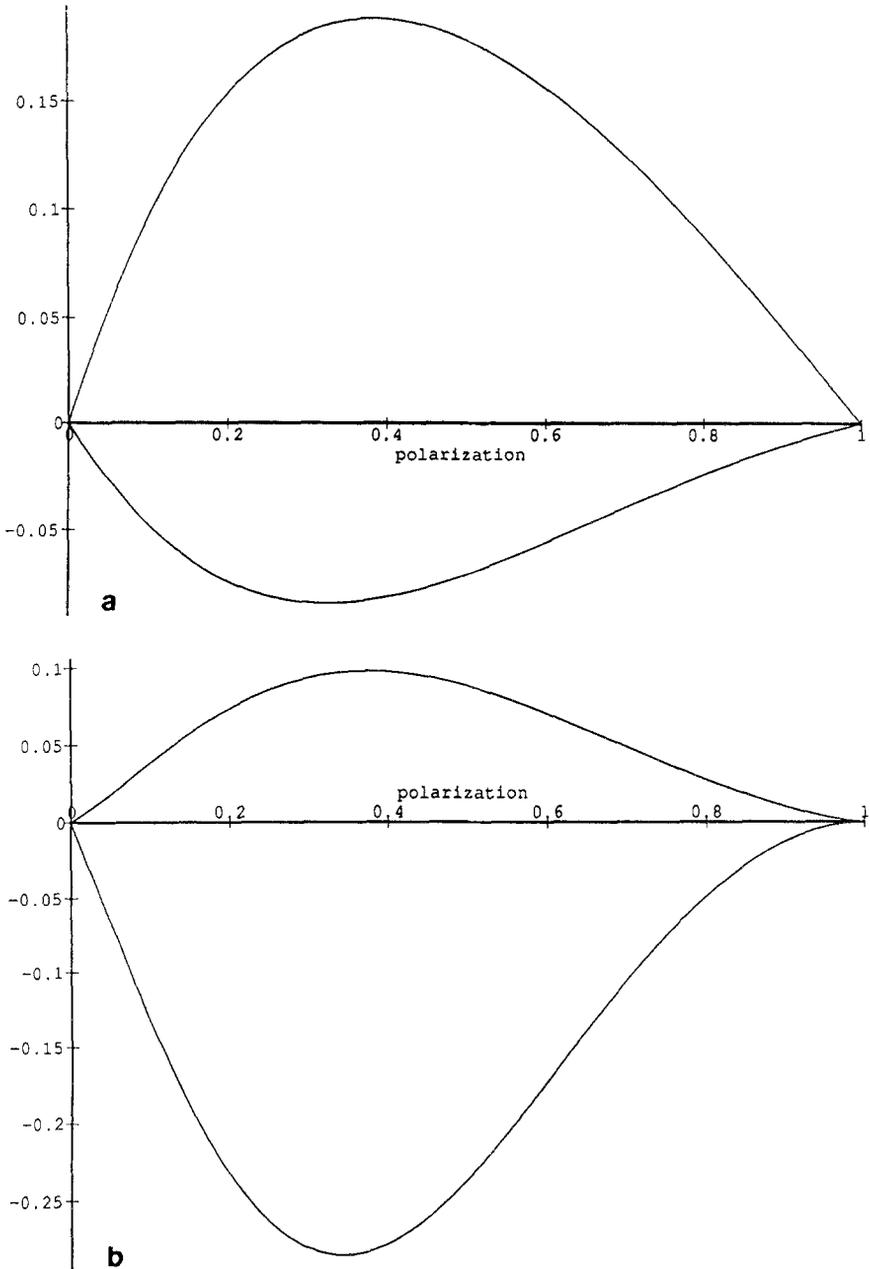


Fig. 4. Polarization dependence of the bounds for α_{21} in quantum gases. (a) functions $f_{5,6}$ for a quantum Boltzmann gas, Eqs. (79) and (76); (b) functions $f_{7,8}$ for degenerate gases, Eqs. (80) and (75).

The coefficient α_{31} has the following high-temperature bounds:

$$f_9 \equiv -(1 + vt^2) < \frac{25m\alpha_{31}}{T^2 N_+ \tau_+^2} < -\frac{125}{1376} \frac{(1 + v)^2 t}{v} (1 - v) \left[\frac{1 - v}{1 + v} - \frac{1 - vt}{1 + vt} \right] \equiv f_{10} \quad (81)$$

Functions $f_{9,10}$ in the quantum region are given in Fig. 5. The bounds for α_{13} can be found from the Onsager relation

$$0 > \alpha_{13} = \frac{4}{TN_+ \tau_+} \frac{\alpha_{31}}{1 + vt} + \frac{5T\tau_+}{172m} \frac{(1 - v^2) t(1 - t)}{(1 + vt)^2}$$

At low temperatures $T \rightarrow 0$ all the heat-related transport coefficients in Fermi systems contain extra powers of T/T_F , and, therefore, are small and not very important. The same is true, as it has been mentioned in Sec. 5, for all slip coefficients α_{ik} which contain the index 3.

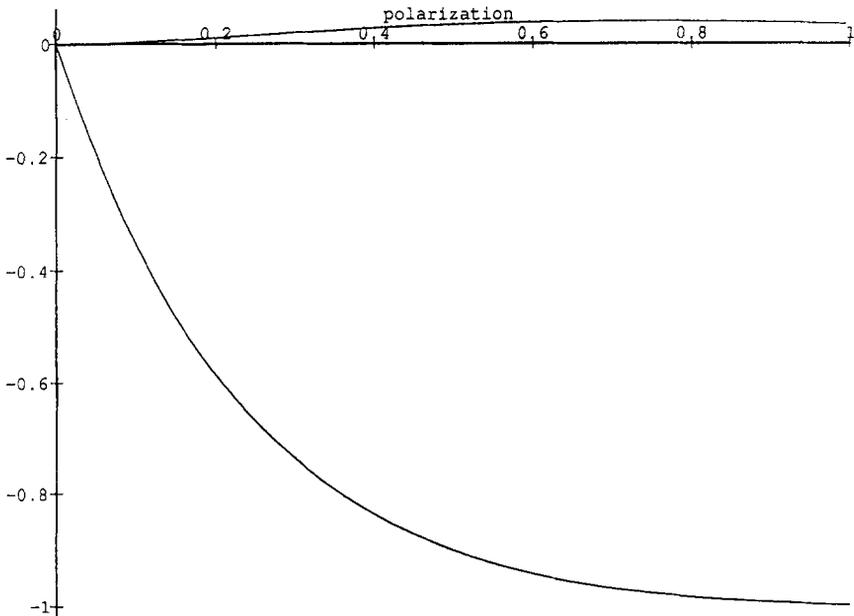


Fig. 5. Polarization dependence of the bounds $f_{9,10}$, Eq. (81), for α_{31} in quantum Boltzmann gases (76).

The second diagonal slip coefficient at high temperatures is restricted by

$$\alpha_{22} \sqrt{\frac{N_+ \tau_+^2}{8} \left(\frac{2mT}{\pi} \right)^{1/2}} > \frac{vt}{1+vt} (1+t) \equiv f_{11} \quad (82)$$

while at low temperatures

$$\alpha_{22} \sqrt{\frac{3m}{16} N_+ \tau_+^2 v_{F+}} > \frac{vt}{1+vt} (1+tu) \equiv f_{12} \quad (83)$$

The polarization dependence of the functions $f_{11,12}$ for quantum gases (75), (76) is given in Fig. 6. The last diagonal coefficient is restricted at high temperatures by the inequality

$$\frac{2\alpha_{33}}{N_+ \tau_+^2} \left(\frac{\pi m}{2T} \right)^{1/2} \frac{m}{T} < \frac{9}{4} (1+vt^2) + \frac{15}{172} \frac{t(1-t)(1-v^2)}{1+vt} \equiv f_{13} \quad (84)$$

(see Fig. 7). At low temperatures this coefficient becomes vanishingly small. The corresponding off-diagonal slip coefficients in the Boltzmann temperature range are

$$\frac{16\alpha_{23}}{N_+ \tau_+^2} \left(\frac{\pi m}{2T} \right)^{1/2} = \left(\frac{\pi m}{2T} \right)^{1/2} \frac{16\alpha_{32}}{TN_+ \tau_+^2} < f_{14}, f_{15} \quad (85)$$

$$f_{14} = 1 - vt^2 + \frac{5}{43} \frac{t(1+t)(1-v^2)}{1+vt}, \quad f_{15} = 12 \frac{vt}{1+vt} (1-t)$$

In the case of quantum gases the functions $f_{14,15}$ intersect (see Fig. 8).

For the sake of comparison, we will add the expressions for the bulk transport coefficients. The diffusion coefficient has the form

$$\rho D \left(\frac{\partial \mu}{\partial c} \right)^{-1} = \frac{m}{2} \frac{N_+ v \tau_+ + \tau_-}{\tau_+ + v \tau_-}, \quad T \gg T_F$$

$$\rho D \left(\frac{\partial \mu}{\partial c} \right)^{-1} = 2m \frac{N_+ v \tau_+ + \tau_-}{\tau_+ + v \tau_-}, \quad T \ll T_F$$

$$k_T \rho D = \frac{5TN_+ \tau_+ + t(1-v^2)}{172} \frac{1}{1+vt}, \quad T \gg T_F$$

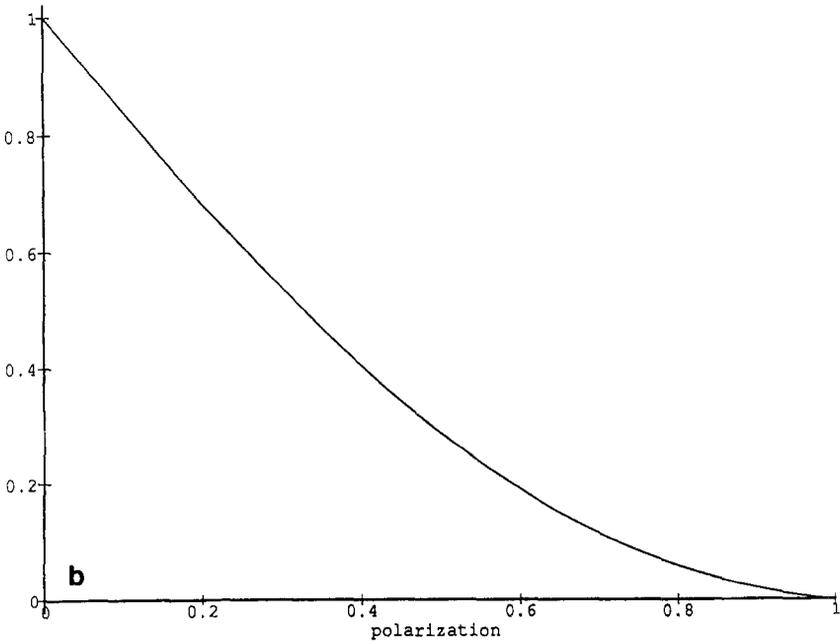
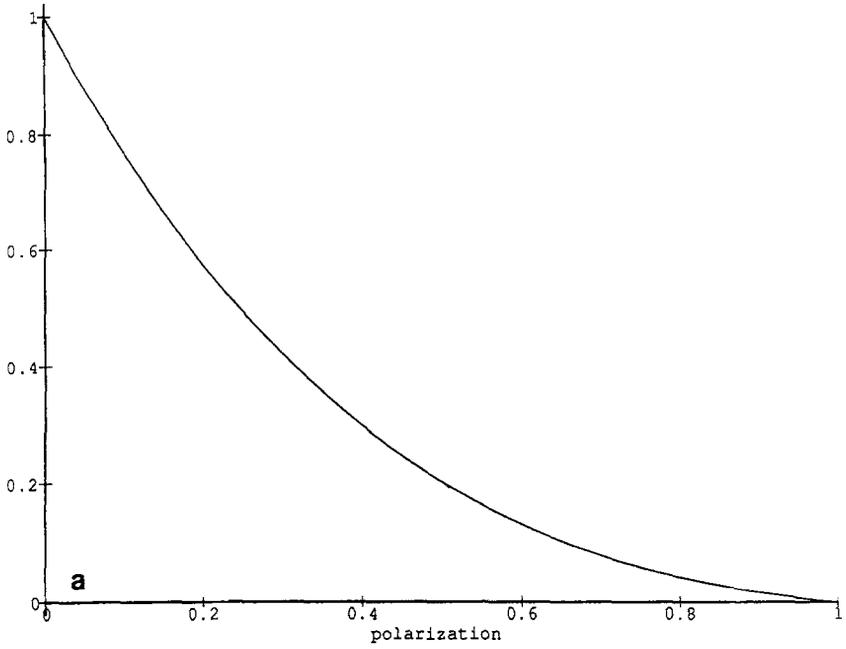


Fig. 6. Bounds for α_{22} in (a) quantum Boltzmann gas, f_{11} , Eqs. (82) and (76), and (b) degenerate gas, f_{12} , Eqs. (83) and (75).

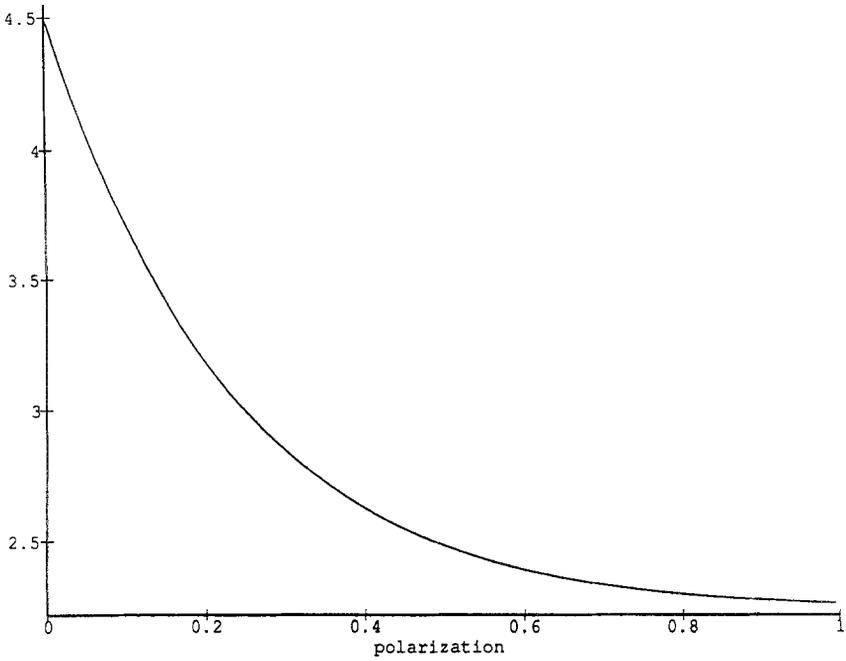


Fig. 7. Polarization dependence of the bound f_{13} for α_{33} , Eq. (84) for a quantum Boltzmann gas (76).

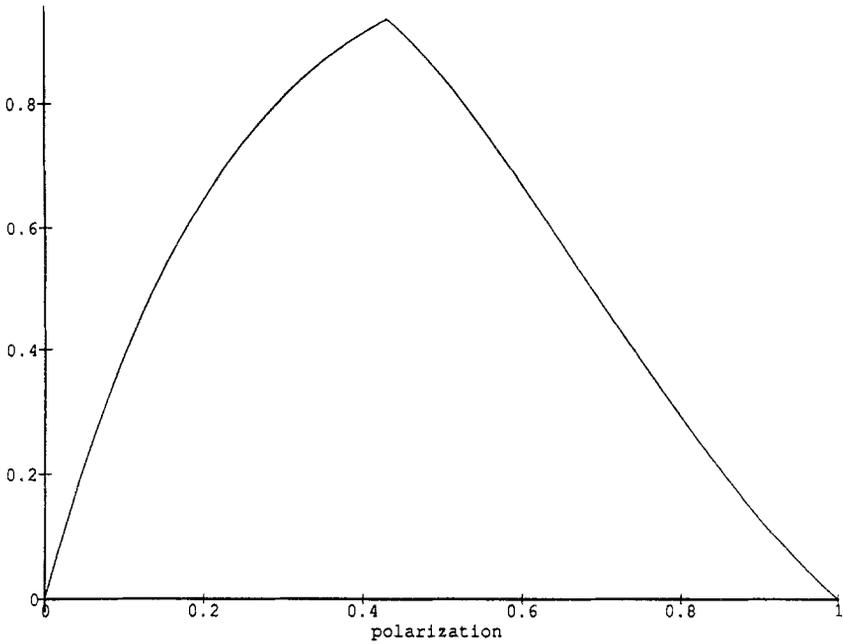


Fig. 8. The best of the bounds $f_{14,15}$ for α_{23} and α_{32} , Eq. (85), for a quantum Boltzmann gas, Eq. (76).

while partial viscosities are given by the following equation:

$$\eta \equiv \eta_+ + \eta_- = \frac{T}{4} (N_+ \tau_+ + N_- \tau_-), \quad T \gg T_F$$

$$\eta \equiv \eta_+ + \eta_- = \frac{m}{5} (N_+ \tau_+ v_{F+}^2 + N_- \tau_- v_{F-}^2), \quad T \ll T_F$$

The values of the slip coefficients in the intermediate temperature range are given by the results of numerical calculations in Appendix B.

APPENDIX A

We will derive the Onsager relations for the slip coefficients in the way similar to the derivation of Waldmann symmetry relations in Ref. 28. As in Ref. 11, we will need the expression for the pressure diffusion coefficient with viscous renormalization. We mentioned in Sec. 2 that our definitions for diffusion currents is slightly different from that of Ref. 11; therefore, the expression for the pressure diffusion ratio is also different.

We start from multiplying the transport equations for two (spin) components of the mixture,

$$\frac{\partial n_{\pm}}{\partial t} + \frac{\partial \varepsilon_{\pm}}{\partial \mathbf{p}} \frac{\partial n_{\pm}}{\partial \mathbf{r}} - \frac{\partial \varepsilon_{\pm}}{\partial \mathbf{r}} \frac{\partial n_{\pm}}{\partial \mathbf{p}} = I_{\text{coll}}^{\pm}(n_+, n_-) \quad (\text{A.1})$$

by $m_{\pm}(\mathbf{v}_{\pm} - \mathbf{u})$ and integrating over momenta (\mathbf{u} is the overall mass velocity). After standard transformations, these two equations reduce to (see Eq. (10) of Ref. 11):

$$N_{\pm} \partial \mu_{\pm} / \partial \mathbf{r} - (\eta_{\pm} / \eta) \partial P / \partial \mathbf{r} = -m_{\pm} \int (\mathbf{v}_{\pm} - \mathbf{u}) I_{\text{coll}}^{\pm} d\Gamma \quad (\text{A.2})$$

The integrals in the r.h.s of Eqs. (A.2) are proportional to the mass currents, \mathbf{j}_{\pm} . Therefore, the bulk diffusion current (3) is proportional to

$$\mathbf{j}_D \equiv \mathbf{j}_+ - \mathbf{j}_- = \text{const} \times \left(N_+ \frac{\partial \mu_+}{\partial \mathbf{r}} - N_- \frac{\partial \mu_-}{\partial \mathbf{r}} - \frac{(\eta_+ - \eta_-)}{\eta} \frac{\partial P}{\partial \mathbf{r}} \right) \quad (\text{A.3})$$

The chemical potentials of the components, μ_{\pm} , should be expressed via μ , Eq. (2), and the pressure P using the identity $dP = N_+ d\mu_+ + N_- d\mu_-$:

$$j_D = \text{const} \times \left(\frac{\partial \mu}{\partial \mathbf{r}} + \frac{\rho}{2\rho_+ \rho_-} \left[\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right] \frac{\partial P}{\partial \mathbf{r}} \right), \quad \rho_{\pm} \equiv m_{\pm} N_{\pm} \quad (\text{A.4})$$

This equation should be compared with the standard definition of diffusion current,

$$j_D = -D\rho \left(\frac{\partial c}{\partial \mathbf{r}} + \frac{k_{sT}}{T} \frac{\partial T}{\partial \mathbf{r}} + \frac{k_{sP}}{P} \frac{\partial P}{\partial \mathbf{r}} \right) \quad (\text{A.5})$$

at constant temperature. The comparison leads to the following expression for the (spin) pressure diffusion ratio (cf. Eq. (12) of Ref. 11):

$$\frac{k_{sP}}{P} = \left(\frac{\partial \mu}{\partial c} \right)_{P,T}^{-1} \left[\left(\frac{\partial \mu}{\partial P} \right)_{c,T} + \frac{\rho}{2\rho_+ \rho_-} \left(\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right) \right] \quad (\text{A.6})$$

Now let us consider two large volumes of polarized gas connected by the tube of the radius R . The mass velocity of gas in the connecting tube is determined by the Poiseuille law (cf. Ref. 11):

$$u_x(r) = -\frac{1}{4\eta} (R^2 - r^2 - 2R\alpha_{11}) \frac{dP}{dx} + \alpha_{12} \frac{d\mu}{dx} + \alpha_{13} \frac{dT}{dx} \quad (\text{A.7})$$

where the pressure P , chemical potential μ , and the temperature T depend only on the coordinate x along the tube. The total heat current through the tube is determined by the bulk heat flow (see Ref. 30) and the surface heat current (5):

$$I_Q = 2\pi \int_0^R q_{\text{bulk}} r dr + 2\pi R Q^s = 2\pi R \left[\frac{\alpha_{31}}{2\eta} R \frac{dP}{dx} + \alpha_{32} \frac{d\mu}{dx} + \alpha_{33} \frac{dT}{dx} \right] - \pi R^2 \kappa \frac{dT}{dx} + \left[k_{sT} \left(\frac{d\mu}{dc} \right)_{P,T} - T \left(\frac{d\mu}{dT} \right)_{P,\alpha} + \mu \right] J_b \quad (\text{A.8})$$

where κ is the thermal conductivity, k_{sT} is the spin thermal diffusion ratio (A.5), and J_b is the bulk part of the total (spin) diffusion current through the tube

$$J_s^{\text{tot}} = -\pi R^2 \left(\frac{\partial \mu}{\partial c} \right)_{P,T}^{-1} \rho D \left[\frac{\rho}{2\rho_+ \rho_-} \left(\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right) \frac{dP}{dx} + \frac{d\mu}{dx} + \left(\frac{\partial \mu}{\partial c} \right)_{P,T} \left(\frac{k_{sT}}{T} + \left(\frac{\partial c}{\partial T} \right)_{\mu,P} \right) \frac{dT}{dx} \right] + 2\pi R \left[\alpha_{21} \frac{R}{2\eta} \frac{dP}{dx} + \alpha_{22} \frac{d\mu}{dx} + \alpha_{23} \frac{dT}{dx} \right] \quad (\text{A.9})$$

The entropy production is given by

$$\dot{S} = -\frac{1}{T} \left\{ J_s^{\text{tot}} \frac{d\mu}{dx} + \int \Pi_{xr} \frac{du_x}{dr} 2\pi r dr + 2\pi R \eta u_x(R) \frac{du_x(R)}{dr} + \frac{I_Q}{T} \frac{dT}{dx} \right\} \quad (\text{A.10})$$

where the stress tensor $\Pi_{xr} = \eta du_x/dr$. Now we have to substitute du_x/dr in Eq. (A.10) by dP/dx with the help of Eq. (A.7):

$$\dot{S} = -\frac{1}{T} \left\{ J_s^{\text{tot}} \frac{d\mu}{dx} + \frac{dP}{dx} \left[\frac{\pi}{\eta} \int \Pi_{xr} r^2 dr + \pi R^2 u_x(R) \right] + \frac{I_Q}{T} \frac{dT}{dx} \right\} \quad (\text{A.11})$$

If we choose

$$X_1 = \frac{dP}{dx}, \quad X_2 = \frac{d\mu}{dx}, \quad X_3 = \frac{dT}{dx} \quad (\text{A.12})$$

as the thermodynamic forces for the entropy production,

$$\dot{S} = -\sum_i \dot{x}_i X_i \quad (\text{A.13})$$

then the thermodynamic velocities are equal to

$$\dot{x}_1 = \frac{\pi}{\eta T} \int r^2 \Pi_{xr} dr + \frac{\pi}{T} u_x(R) R^2, \quad \dot{x}_2 = \frac{1}{T} I_D, \quad \dot{x}_3 = \frac{1}{T^2} I_Q \quad (\text{A.14})$$

The thermodynamic velocities (A.14) are themselves the linear combinations of the thermodynamic forces,

$$\dot{x}_i = \gamma_{ik} X_k \quad (\text{A.15})$$

According to the Onsager principle, $\gamma_{ik} = \gamma_{ki}$. As a result, one immediately recovers Eq. (38),

$$\alpha_{21} = \eta \alpha_{12} + \left(\frac{\partial \mu}{\partial c} \right)^{-1} \frac{\rho^2 D \eta}{2\rho_+ \rho_-} \left[\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right] \quad (\text{A.16})$$

and the relation between α_{13} and α_{31} :

$$\begin{aligned} \alpha_{32} &= T \alpha_{23} \\ \alpha_{31} &= \eta T \alpha_{13} - \frac{k_{sT} (\partial \mu / \partial c)_{P,c} - T (\partial \mu / \partial T)_{P,c} + \mu}{(\partial \mu / \partial c)_{P,T}} \\ &\quad \times \frac{D \rho^2 \eta}{2\rho_+ \rho_-} \left[\frac{\rho_+ - \rho_-}{\rho} - \frac{\eta_+ - \eta_-}{\eta} \right] \end{aligned} \quad (\text{A.17})$$

APPENDIX B

We are interested in the values of the functions $L_n(z)$, $K_n(z)$, $M_n(z)$, $Q_n(z)$, $V_n(z)$, $G_n(z)$, $S_n(z)$, Eqs. (15), (27), (40), (46), and (66) at $z=0$ at different temperatures and spin polarizations. These functions can be conveniently parametrized in the following way:

$$\begin{aligned} \tilde{\mu}_{\pm} &= \frac{\mu_{\pm}}{T_F}, & \tilde{T} &= \frac{T}{T_F}, & T_F &= \frac{\hbar^2}{2m} (3\pi^2 N)^{2/3} \\ C_{\pm n} &= \frac{3}{16} \frac{\tau_{\pm\pm}^{n-1}}{m_{\pm}^n} N, & A_{\pm n} &= (2m_{\pm})^{(n+2)/2} T_F^{n/2} \tilde{T}^{(n+3)/2} \\ I_{\pm n} &= \int_0^1 (-\ln t)^{(n+3)/2} \frac{dt e^{\tilde{\mu}_{\pm}/\tilde{T}}}{(t e^{\tilde{\mu}_{\pm}/\tilde{T}} + 1)^2} \end{aligned} \quad (\text{B.1})$$

where T_0 is the degeneracy temperature of a non-polarized gas. In these notations

$$\begin{aligned} L_{\pm n}(0) &= \left(\frac{1}{n+1} - \frac{1}{n+3} \right) C_{\pm n} A_{\pm n} I_{\pm n} \\ K_{\pm n}(0) &= \left(\frac{1}{n} - \frac{1}{n+2} \right) m_{\pm} C_{\pm n} A_{\pm n-1} I_{\pm n-1} \\ M_{\pm n}(0) &= \left(\frac{1}{n} - \frac{1}{n+2} \right) \frac{1}{m_{\pm}} C_{\pm n} A_{\pm n+1} I_{\pm n+1} \\ Q_{\pm n}(0) &= \left(\frac{1}{n-1} - \frac{1}{n+1} \right) m_{\pm} C_{\pm n} A_{\pm n-2} I_{\pm n-2} \\ V_{\pm n}(0) &= \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \frac{1}{m_{\pm}} C_{\pm n} A_{\pm n} I_{\pm n}, & G_{\pm n}(0) &= \frac{m_{\pm}}{2} V_{\pm n}(0) \\ S_{\pm n}(0) &= \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \frac{1}{2m_{\pm}^2} C_{\pm n} A_{\pm n+2} I_{\pm n+2} \end{aligned} \quad (\text{B.2})$$

Integrals $I_{\pm n}$ with $n=1, 2, 3, 4, 5$, which enter the expressions for the slip coefficients α_{ik} , are plotted in Figs. 9(a)–(f) for several values of spin polarization. We will also give the analytic values of all functions in the limiting case of Boltzmann regime $T \gg T_F$,

$$\begin{aligned} K_1^{\pm}(0) &= \frac{1}{8} m_{\pm} N_{\pm}; & K_2^{\pm}(0) &= \frac{1}{8} \left(\frac{2Tm_{\pm}}{\pi} \right)^{1/2} N_{\pm} \tau_{\pm} \\ K_3^{\pm}(0) &= \frac{1}{8} TN_{\pm} \tau_{\pm}^2 \end{aligned}$$

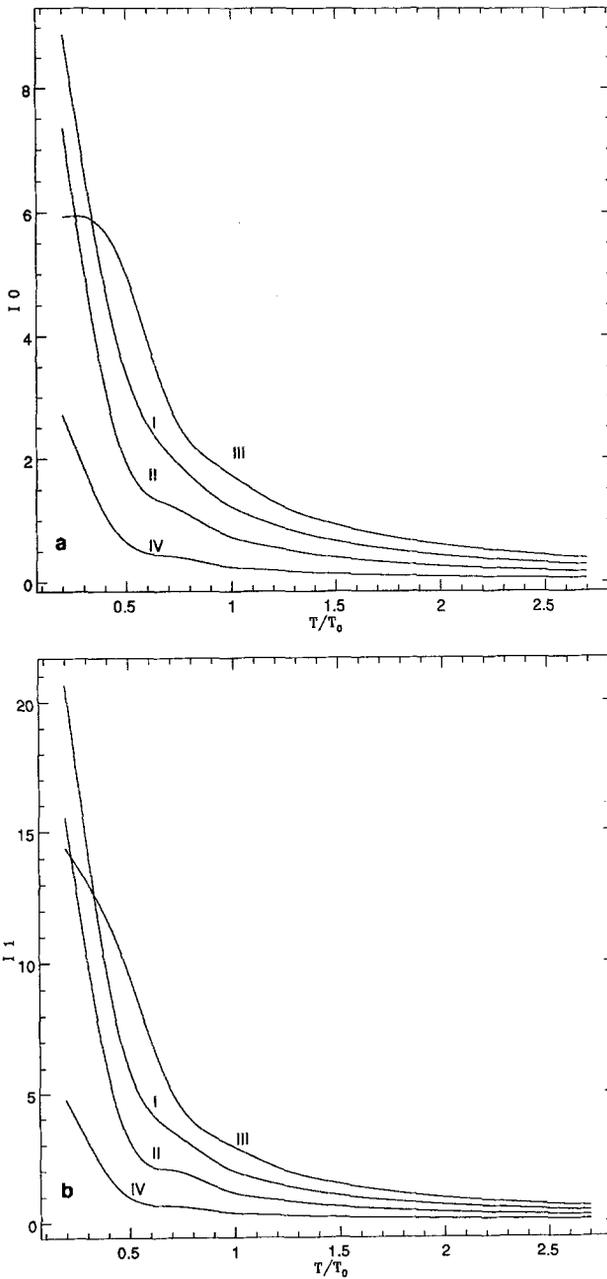


Fig. 9. Integrals $I_{\pm n}$ (B.1) as a function of temperature for different n and two different concentrations (polarizations). (a) I_0 , (b) I_1 , (c) I_2 , (d) I_3 , (e) I_4 , (f) I_5 . Curves: I — I_+ ($c=0.25$), II — I_- ($c=0.25$), III — I_+ ($c=0.75$), IV — I_- ($c=0.75$).

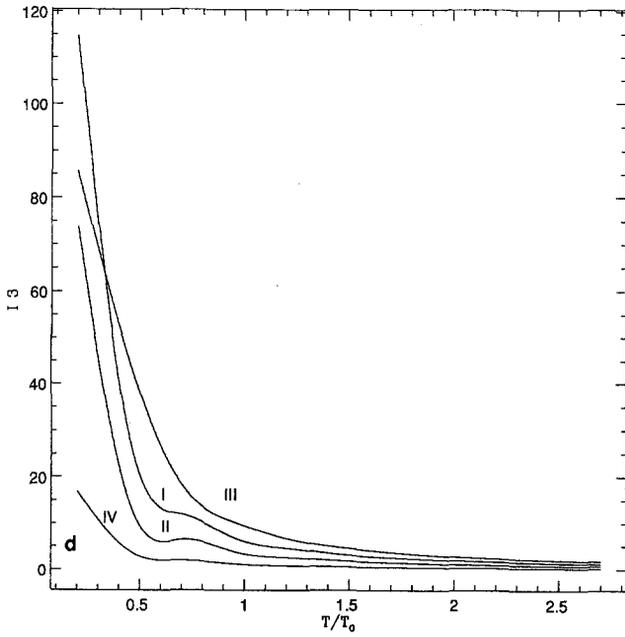
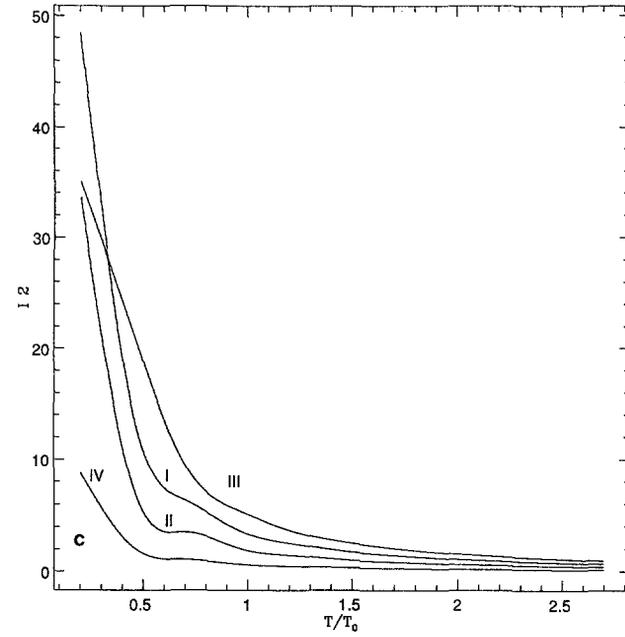


Fig. 9. Continued.

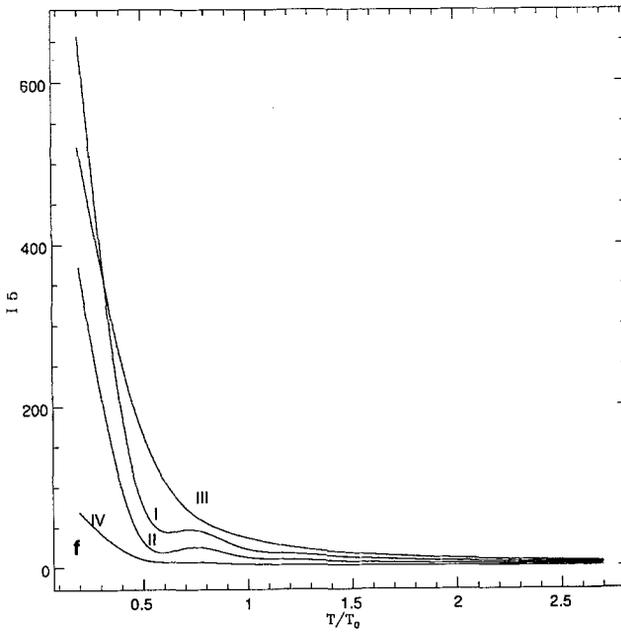
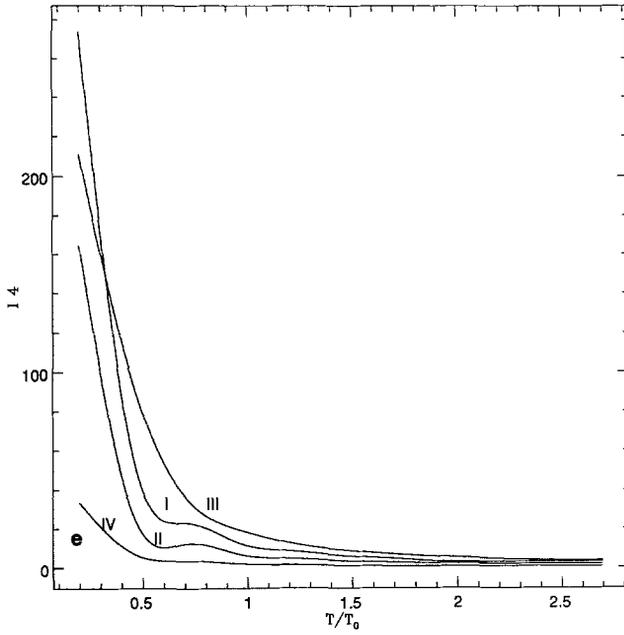


Fig. 9. Continued.

$$\begin{aligned}
L_1^\pm(0) &= \frac{1}{8} \left(\frac{2Tm_\pm}{\pi} \right)^{1/2} N_\pm; & L_2^\pm(0) &= \frac{1}{8} TN_\pm \tau_\pm \\
L_3^\pm(0) &= \frac{T^{3/2}}{(8\pi m_\pm)^{1/2}} N_\pm \tau_\pm^2 \\
M_1^\pm(0) &= \frac{5}{8} TN_\pm; & M_2^\pm(0) &= \frac{3}{4} \frac{T^{3/2}}{(\pi m_\pm/2)^{1/2}} N_\pm \tau_\pm \\
M_3^\pm(0) &= \frac{7}{8} \frac{T^2}{m_\pm} N_\pm \tau_\pm^2 \\
Q_2^\pm(0) &= \frac{1}{8} N_\pm \tau_\pm; & Q_3^\pm(0) &= \frac{1}{8} \left(\frac{2T}{\pi m_\pm} \right)^{1/2} N_\pm \tau_\pm^2 \\
V_2^\pm(0) &= \frac{5}{8} \frac{T}{m_\pm} N_\pm \tau_\pm; & V_3^\pm(0) &= \frac{3}{4} (2/\pi)^{1/2} \left(\frac{T}{m_\pm} \right)^{3/2} N_\pm \tau_\pm^2 \\
G_2^\pm(0) &= \frac{5}{16} TN_\pm \tau_\pm; & G_3^\pm(0) &= \frac{3}{8} (2/\pi m_\pm)^{1/2} T^{3/2} N_\pm \tau_\pm^2 \\
S_2^\pm(0) &= \frac{35}{16} \frac{1}{m_\pm} T^2 N_\pm \tau_\pm; & S_3^\pm(0) &= 3(2/\pi)^{1/2} \frac{T^{5/2}}{m_\pm^{3/2}} N_\pm \tau_\pm^2
\end{aligned}$$

and at low temperatures, when the system is degenerate, $T \ll T_F$

$$\begin{aligned}
L_n^\pm(0) &= \frac{3}{4} \left(\frac{1}{n+1} - \frac{1}{n+3} \right) \tau_\pm^{n-1} m_\pm N_\pm v_{F\pm}^n \\
K_n^\pm(0) &= \frac{3}{4} \left(\frac{1}{n} - \frac{1}{n+2} \right) \tau_\pm^{n-1} m_\pm N_\pm v_{F\pm}^{n-1} \\
M_n^\pm(0) &= \frac{3}{4} \left(\frac{1}{n} - \frac{1}{n+2} \right) \tau_\pm^{n-1} m_\pm N_\pm v_{F\pm}^{n+1} \\
Q_{(n>1)}^\pm(0) &= \frac{3}{4} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \tau_\pm^{n-1} N_\pm v_{F\pm}^{n-2} \\
V_{(n>1)}^\pm(0) &= \frac{3}{4} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \tau_\pm^{n-1} N_\pm v_{F\pm}^n \\
G_{(n>1)}^\pm(0) &= \frac{3}{8} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \tau_\pm^{n-1} m_\pm N_\pm v_{F\pm}^n \\
S_{(n>1)}^\pm(0) &= \frac{3}{8} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \tau_\pm^{n-1} m_\pm N_\pm v_{F\pm}^{n+2}
\end{aligned}$$

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