Consider the Ornstein-Uhlenbeck process as specified by the Fokker-Planck equation,
\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial x} (\kappa x P) + \frac{1}{2} \gamma \frac{\partial^2 P}{\partial x^2}, \]
for the conditional probability distribution \( P(x, t|x_0) \), where \( x_0 \) specifies the initial value of all sample paths: \( P(x, 0|x_0) = \delta(x - x_0) \).

(a) Derive from the 2nd order PDE (1) the 1st order PDE for the characteristic function:
\[ \frac{\partial \Phi}{\partial t} + \kappa s \frac{\partial}{\partial s} \Phi(s, t) = -\frac{1}{2} \gamma s^2 \Phi(s, t), \quad \Phi(s, t) = \int_{-\infty}^{+\infty} dx e^{isx} P(x, t|x_0). \]

(b) Solve (2) by the method of characteristics,
\[ \frac{1}{dt} = \frac{\kappa s}{ds} = -\frac{1}{2} \gamma s^2 \Phi \frac{d}{\Phi}. \]

(c) Infer from the solution \( \Phi(s, t) \) an explicit expression for \( P(x, t|x_0) \).

Solution: